Dynamique des populations

Construction et applications à la longévité

Alexandre Boumezoued

Supervision of, and joint work with, Nicole El Karoui (LPMA, Paris 6) and Stéphane Loisel (ISFA, Lyon 1)
Paris 6 University

Probability and Random Models Laboratory, UMR-CNRS 7599

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Aim of this talk

- Construction: from Poisson to population process
- Example 1: "Cohort effect" in insurance

Plan

Microscopic models

Microscopic models in many fields:

- Agent-based models in economics (Orcutt, 1957)
- Microsimulation models of government bodies (Ex : INSEE, model "DESTINIE")
- Individual-Based models in ecology (mathematical framework)
  - Modelling a population with birth, death & mutation at birth
  - Population structured by traits (*i.e.* individual characteristics) (Fournier-Méléard 2004, Champagnat-Ferrière-Méléard 2006)
  - Extension to age-structured populations (Tran 2006, Ferrière-Tran 2009)
- First formulation and tests for human populations: Bensusan Phd thesis (2010)

Marked Poisson point process

- Consider $\bar{T}_1 < \bar{T}_2 < ... < \bar{T}_n < ...$ the ordered times of jump of a Poisson process $\bar{N}$ with parameter $\bar{\lambda}$.
    - That is, $(\bar{T}_{n+1} - \bar{T}_n)$ are iid $\sim \mathcal{E}xp(\bar{\lambda})$.
- Then "mark" each time of jump $\bar{T}_n$ with an independent r.v. Y_n drawn on a space E with a probability density $\bar{\mu}(\mathrm{d}y)$, the sequence (Y_n) being independent of the $(\bar{T}_n)$.
- Such process is called Marked Poisson point process on $\mathbb{R}_+ \times E$ with intensity measure $q(\mathrm{d}s, \mathrm{d}y) = \bar{\lambda}\mathrm{d}s\bar{\mu}(\mathrm{d}y)$.

The associated random measure

- It can be represented as a random point measure on $\mathbb{R}_+ \times E$, say

$$Q(\mathrm{d}s, \mathrm{d}y) = \sum_{n \geq 1} \delta_{(\bar{T}_n, Y_n)}(\mathrm{d}s, \mathrm{d}y).$$

- Note that the original Poisson process can be recovered by

$$\bar{N}_t = \sum_{n \geq 1} \mathbf{1}_{\bar{T}_n \leq t} = \sum_{n \geq 1} \mathbf{1}_{\bar{T}_n \leq t} \mathbf{1}_{Y_n \in E} = \int_0^t \int_E Q(\mathrm{d}s, \mathrm{d}y).$$

Martingale property

- Denote $(\mathcal{G}_t)$ the filtration generated by Q. The following martingale property holds: for each bounded f, $\langle Q, f\rangle_t - \langle q, f\rangle_t$ is a $(\mathcal{G}_t)-$martingale where we denote

$$\langle Q, f\rangle_t = \int_0^t \int_E f(s,y) Q(\mathrm{d}s, \mathrm{d}y) = \sum_{n\geq 0} f(T_n, Y_n). \qquad (1)$$

- The same property holds if $f(s,y)$ is $(\mathcal{G}_t)-$predictable, provided that $\mathbb{E}[\langle q, |f|\rangle] < +\infty$.

Powerful representation/simulation tool

- Consider a non-homogenous Poisson process with (deterministic) intensity λ_t
  - that is a counting process with the property that $N_t - \int_0^t \lambda_s \mathrm{d}s$ is a martingale.
- How to represent this process ?
- $\frac{\lambda_t}{\bar{\lambda}} \bar{N}_t$ is a process with the right intensity, but not counting anymore...

Powerful representation/simulation tool

- Assume that $\lambda_s \leq \bar{\lambda}$.
- The key idea is to provide additional information: consider a marked Poisson point process with iid marks Θ_n uniformly distributed in $[0,1]$: $Q(\mathrm{d}s, \mathrm{d}\theta)$ has intensity measure $q(\mathrm{d}s, \mathrm{d}\theta) = \bar{\lambda}\mathrm{d}s\mathrm{d}\theta$ on $\mathbb{R}_+ \times [0,1]$.
- $\lambda_s \mathrm{d}s = \int_{[0,1]} \mathbf{1}_{\theta \leq \lambda_s/\bar{\lambda}} \bar{\lambda} \mathrm{d}\theta \Rightarrow$ preserve counting

$$N_t := \sum_{n \geq 1} \mathbf{1}_{\bar{T}_n \leq t} \mathbf{1}_{\Theta_n \leq \lambda_{\bar{T}_n}/\bar{\lambda}} = \int_0^t \int_{[0,1]} \mathbf{1}_{\theta \leq \lambda_s/\bar{\lambda}} Q(\mathrm{d}s, \mathrm{d}\theta).$$

$\Rightarrow$ N is a counting process with the desired compensator (martingale property) [Same result if (λ_t) is $(\mathcal{G}_t)$-predictable]

Non homogenous process simulation

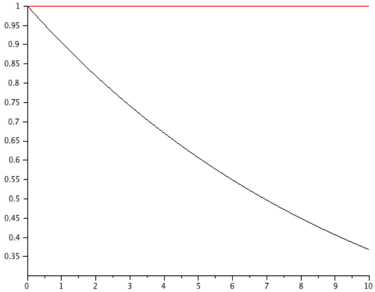
Simulation by Thinning of a counting process with rate λ_t, assuming that $\lambda_t \leq \overline{\lambda}$

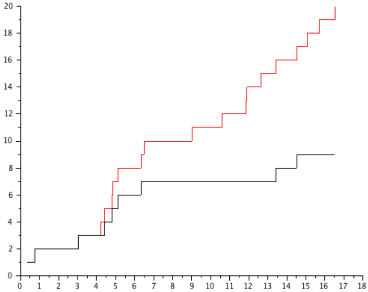
- Let $(\bar{T}_n)$ the jump times of a Poisson process $\overline{N}_t$ with rate $\overline{\lambda}$:

$$\bar{T}_{n+1} - \bar{T}_n \sim \mathcal{E}xp\left(\bar{\lambda}\right)$$

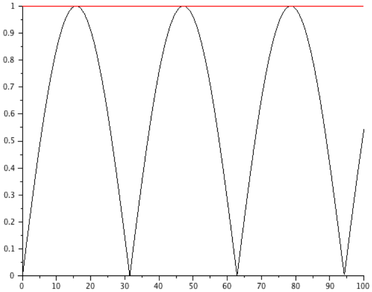
- Recursively, at time $\bar{T}_n$ pick a Bernouilli independent r.v. U_n s.t. $\mathbb{P}(U_n = 1) = \lambda_{\bar{T}_n}/\overline{\lambda}$.
- Then $N_t := \mathrm{cardinal}\{k : U_k = 1, \bar{T}_k \leq t\}$ is a counting process with rate λ_t
- $(\bar{T}_n)$ are interpreted as candidate times for the system
- The thinning method makes easier the simulation of counting processes with complex rates

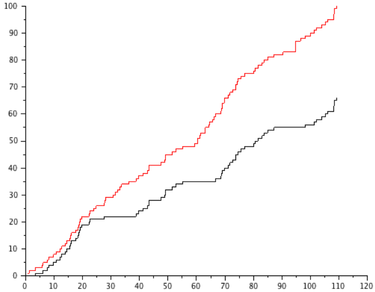
Poisson process by thinning 1





Poisson process by thinning 2





Unbounded intensities

- How do we manage the representation if the predictable process λ_t is not bounded ?
- this requires that the space of marks θ is $\mathbb{R}_+$ embedded with Lebesgue measure
- the Poisson point measure $Q(\mathrm{d}s, \mathrm{d}y)$ with intensity measure $q(\mathrm{d}s, \mathrm{d}y) = \mathrm{d}s\mu(\mathrm{d}y)$ on $\mathbb{R}_+ \times E$, with μ (only) sigma-finite, is defined as the random measure taking values in $\mathbb{N} \cup \{+\infty\}$verifying

  (i) for all non-overlapping measurable sets $B_1, ..., B_k$ of $\bar{E} = \mathbb{R}_+ \times E$, r.v. $Q(B_1), ..., Q(B_k)$ are independent ,

  (ii) for all measurable set $B \subset \bar{E}$ such that $q(B) < +\infty$, $Q(B) \sim \mathcal{P}oisson\,(q(B))$.

  (iii) $Q(\{0\} \times E) = 0$ (this ensures no jump at time 0)

Martingale property

- The martingale property still holds
- This allows to construct a counting process with $\mathcal{G}_t$-predictable intensity λ_t such that $\int_0^t \lambda_s \mathrm{d}s < +\infty$ a.s..

$$N_t := \sum_{n\geq 1} \mathbf{1}_{S_n \leq t} \mathbf{1}_{S_n \leq \lambda_{S_n}} = \int_0^t \int_{\mathbb{R}_+} \mathbf{1}_{\theta \leq \lambda_s} Q(\mathrm{d}s, \mathrm{d}\theta). \qquad (2)$$

$\Rightarrow$ N is a counting process s.t. $N_t - \int_0^t \lambda_s \mathrm{d}s$ is a $(\mathcal{G}_t)-$local martingale

The pure death process

- Consider a death process starting with Z_0 individuals
- Each individual has death rate $\bar{d}$ (its lifetime $\tau \sim \mathcal{E}xp(\bar{d})$).
- The Z_0 individuals are independent
  $\Rightarrow$ First time of death $\sim \mathcal{E}xp(\bar{d} Z_0)$
  $\Rightarrow$ Intensity at time t: $\lambda_t := \bar{d} Z_{t-}$
- The pure death process is defined as the solution to the stochastic differential equation

$$Z_t = Z_0 - \int_0^t \int_{\mathbb{R}_+} \mathbf{1}_{\theta \leq \bar{d} Z_{s-}} Q(\mathrm{d}s, \mathrm{d}\theta).$$

[Existence and uniqueness in the general case]

Birth-death process

- Pure birth process defined as the solution to

$$Z_t = Z_0 + \int_0^t \int_{\mathbb{R}_+} \mathbf{1}_{\theta \leq \bar{b} Z_{s-}} Q(\mathrm{d}s, \mathrm{d}\theta).$$

- Birth-death process defined as the solution to

$$Z_t = N_0 + \int_0^t \int_{\mathbb{R}_+} \left(\mathbf{1}_{\theta \leq \bar{b} Z_{s-}} - \mathbf{1}_{\bar{b} Z_{s-} < \theta \leq (\bar{b}+\bar{d}) Z_{s-}} \right) Q(\mathrm{d}s, \mathrm{d}\theta).$$

- Dynamics of the birth-death process:

$$\mathrm{d}Z_t = Z(\mathrm{d}t) = \int_{\mathbb{R}_+} \left(\mathbf{1}_{\theta \leq \bar{b} Z_{t-}} - \mathbf{1}_{\bar{b} Z_{t-} < \theta \leq (\bar{b}+\bar{d}) Z_{t-}} \right) Q(\mathrm{d}t, \mathrm{d}\theta),$$

$\Rightarrow Z(\mathrm{d}t)$ is a signed point measure on $\mathbb{R}_+$.

Pure birth process with age

- Add an age to each individual: if born at time t_0 the individual age $t_1 - t_0$ at time t_1
- Consider a measure on $\mathbb{R}_+$ keeping track of all ages in the population: $Z_t(\mathrm{d}a) = \sum_{i=1}^{L_t} \delta_{A_i(Z_t)}(\mathrm{d}a)$ where $L_t = \langle Z_t, \mathbf{1}\rangle$
- The (measure-valued) birth process $(Z_t(\mathrm{d}a))$ is defined as the solution to the following equation:

$$Z_t(\mathrm{d}a) = Z_0^t(\mathrm{d}a) + \int_0^t \int_{\mathbb{R}_+} \delta_{t-s}(\mathrm{d}a)\mathbf{1}_{\theta \leq \bar{b}\langle Z_{s-}, \mathbf{1}\rangle} Q(\mathrm{d}s, \mathrm{d}\theta).$$

For any continuous and bounded function $f(a)$, one has

$$\langle Z_t, f\rangle = \langle Z_0^t, f\rangle + \int_0^t \int_{\mathbb{R}_+} f(t-s)\mathbf{1}_{\theta \leq \bar{b}\langle Z_{s-}, \mathbf{1}\rangle} Q(\mathrm{d}s, \mathrm{d}\theta).$$

Birth process with age dependent birth rate

- Each individual with age a has a birth rate $b(a)$ such that $b(a) \leq \bar{b}$.
- Intensity $\lambda_t := \sum_{i=1}^{\langle Z_{t-}, \mathbf{1} \rangle} b(A_i(Z_{t-})) = \langle Z_{t-}, b \rangle$.
- The process is the solution to the equation

$$Z_t(\mathrm{d}a) = Z_0^t(\mathrm{d}a) + \int_0^t \int_{\mathbb{R}_+} \delta_{t-s}(\mathrm{d}a) \mathbf{1}_{\theta \leq \langle Z_{s-}, b \rangle} Q(\mathrm{d}s, \mathrm{d}\theta).$$

Equivalent formulation

- Specific linear form: $\lambda_t = \sum_{i=1}^{\langle Z_{t-}, \mathbf{1}\rangle} b(A_i(Z_{t-}))$
- let Q be a PPM with intensity measure $q(\mathrm{d}s, \mathrm{d}i, \mathrm{d}\theta) = \mathrm{d}s n(\mathrm{d}i)\mathrm{d}\theta$ on $\mathbb{R}_+ \times \mathbb{N}^* \times \mathbb{R}_+$, where n is the counting measure on $\mathbb{N}^*$, that is for any $A \subset \mathbb{N}^*$, $n(A)$ is the number of elements in A.
- Then the previous birth process with age can also be defined as the solution of the equation

$$
\begin{aligned}
Z_t(\mathrm{d}a) &= Z_0^t(\mathrm{d}a) \\
&+ \int_0^t \int_{\mathbb{N}^* \times \mathbb{R}_+} \delta_{t-s}(\mathrm{d}a) \mathbf{1}_{i \leq \langle Z_{s-}, \mathbf{1}\rangle} \mathbf{1}_{\theta \leq b(A_i(Z_{s-}))} Q(\mathrm{d}s, \mathrm{d}i, \mathrm{d}\theta).
\end{aligned}
$$

Including characteristics

- Construction of a random point measure $\Gamma(\mathrm{d}s, \mathrm{d}y)$ with general intensity measure $\gamma(\mathrm{d}s, \mathrm{d}y)$
- Assume that it admits a density: $\gamma(\mathrm{d}s, \mathrm{d}y) = \gamma(s, y)\, \mathrm{d}s\, \mu(\mathrm{d}y)$
- Let $Q(\mathrm{d}s, \mathrm{d}\theta, \mathrm{d}x')$ be a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{X}$ with intensity measure $\mathrm{d}s\mathrm{d}\theta\mu(\mathrm{d}y)$ and natural filtration $(\mathcal{G}_t)$
- Assume that $\gamma(s, y)$ is $(\mathcal{G}_t)$-predictable and $\int_0^t \int_E \gamma(s, y)\mathrm{d}s\mu(\mathrm{d}y) < +\infty$ a.s.
- Then
$$\Gamma(\mathrm{d}s, \mathrm{d}y) = \int_{\mathbb{R}_+} \mathbf{1}_{0 \leq \theta \leq \gamma(s,y)} Q(\mathrm{d}s, \mathrm{d}y, \mathrm{d}\theta)$$
has intensity measure $\gamma(s, y)$.

General model: birth-death-swap

- Let (Y_t) be a cadlag stochastic process ("random environment"), independent of Q

Demographic rates: an individual of characteristics $x_t \in \mathcal{X} \subset \mathbb{R}^d$ and age $a_t \in [0, \bar{a}]$ at time t,

- Dies at rate $d\,(x_t, a_t, t, Y)$
  - $\mathbb{P}(T_{death} \geq t \mid Y) = \exp\left(-\int_0^t d(x_s, a_s, s, Y)\mathrm{d}s\right)$ (born at 0)
- Gives birth at rate $b\,(x_t, a_t, t, Y)$ and the new individual has traits $x' \sim k^b(x_t, a_t, t, x')\gamma(\mathrm{d}x')$
- Evolves during life (swap) at rate $e\,(x_t, a_t, t, Y)\,\gamma(\mathrm{d}x')$ from traits x_t to $x' \sim k^e(x_t, a_t, t, x')\gamma(\mathrm{d}x')$

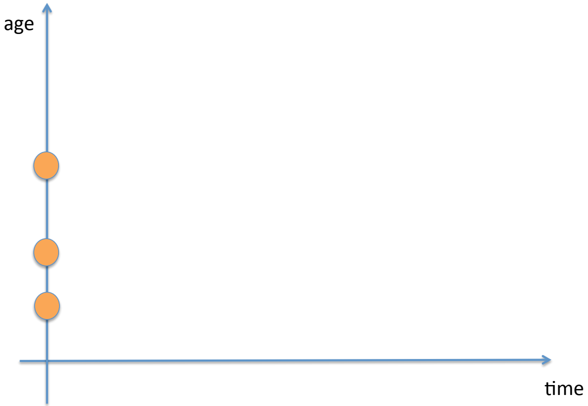
Simulation algorithm

- Assumption of bounded demographic rates with $\bar{b}, \bar{d}$ and $\bar{e}$
  $\Rightarrow$ use of the Thinning method ($\sim$ Inspection times)

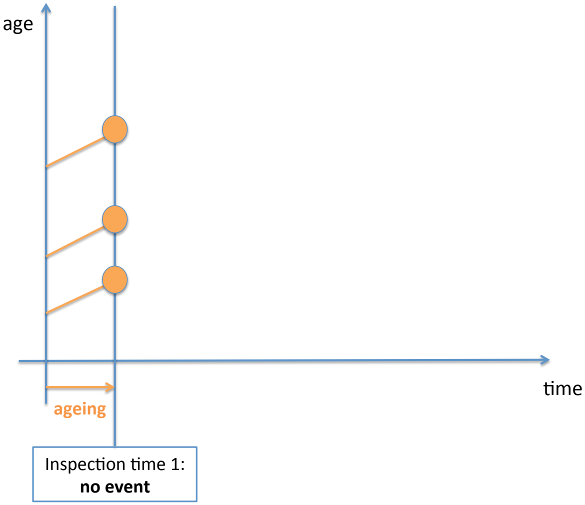
1. Start with N indiv. at T, generate $\tau \sim \mathcal{E}xp\left(N(\bar{d}+\bar{b}+\bar{e})\right)$
   - The bigger the population, the more it is inspected
2. Select an individual (x^I, a^I) uniformly and compute:
   $p_1 = \frac{b(x^I,a^I,T+\tau,Y)}{\bar{b}+\bar{d}+\bar{e}}$, $p_2 = \frac{d(x^I,a^I,T+\tau,Y)}{\bar{b}+\bar{d}+\bar{e}}$, $p_3 = \frac{e(x^I,a^I,T+\tau,Y)}{\bar{b}+\bar{d}+\bar{e}}$
   - Only one individual is checked (not exhaustive)
3. Determine the nature of inspection at time $T+\tau$

- Birth: add a new individual with probability p_1
- Death: remove (x^I, a^I) with probability p_2
- Evolution: change traits of (x^I, a^I) with probability p_3
- No event with probability $1 - p_1 - p_2 - p_3$

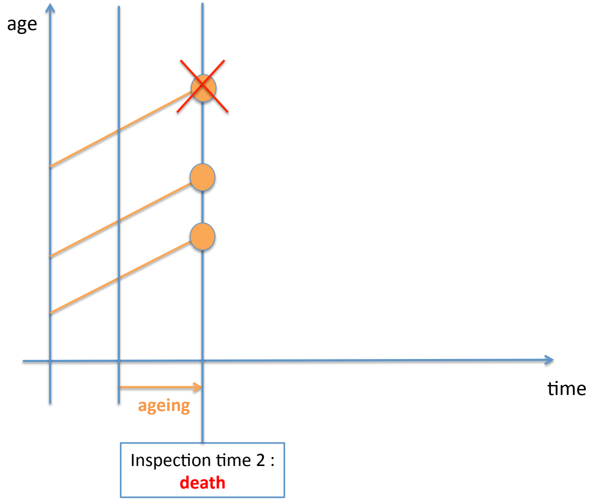
Simulation algorithm



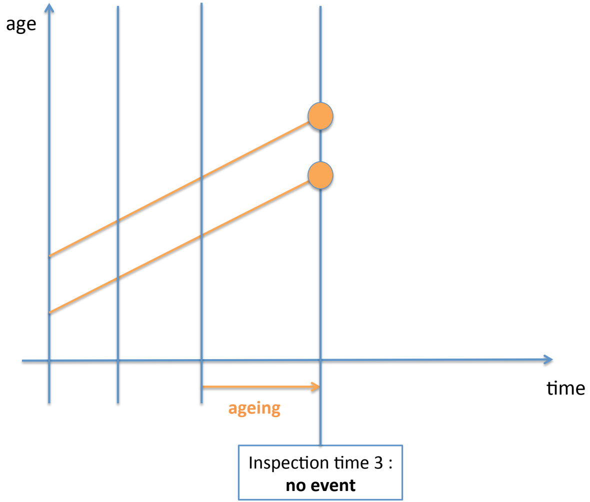
Simulation algorithm



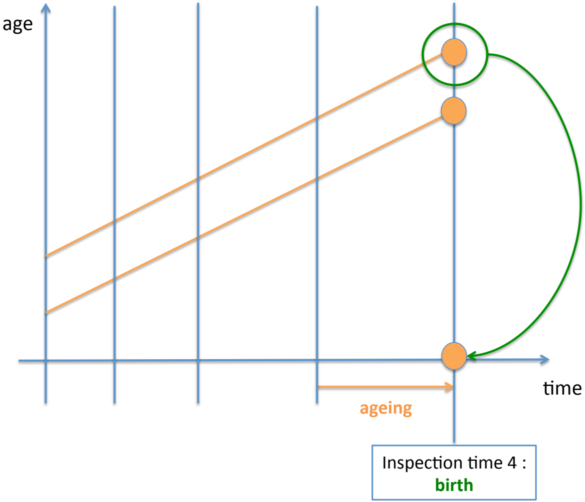
Simulation algorithm



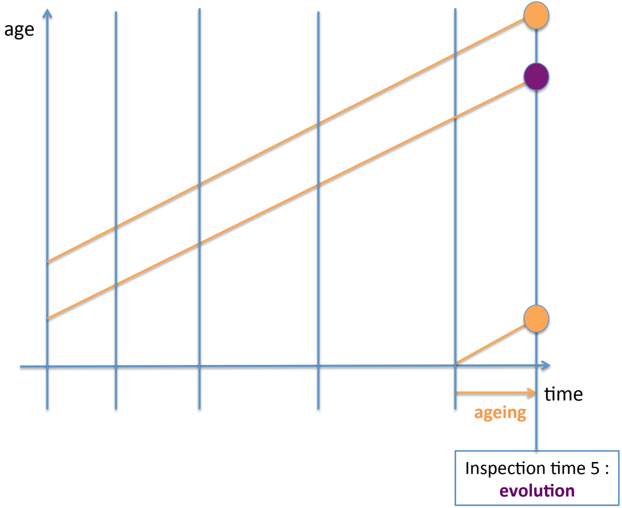
Simulation algorithm



Simulation algorithm



Simulation algorithm



Plan

Cohort effect: framework

- The largest data sets: national basis (see e.g. Human Mortality database), by age and gender
- Used for decision making for (public and private) pension systems
- In particular: possible **hedging with standard longevity indices**
- Ex: standard longevity bond
  $\Rightarrow$ pays a coupon proportional to the number of survivors in a predetermined population (ex: people born in 1960 in UK)
- Ex: standard longevity swap
  $\Rightarrow$ the insurer exchanges a fixed nominal term structure (expected survival of its portfolio) against the reference survival index (ex: UK) payed by the investor

- Problem: the link between the mortality of the insurer's portfolio and that of the national population is not stable (basis risk)
- The heterogeneity of the national reference population is not taken into account $\Rightarrow$ can be crucial to understand variations of reference survival indicators
- Aim: show how heterogenous birth patterns can create artificial mortality improvement for a reference population

Cohort effect, I

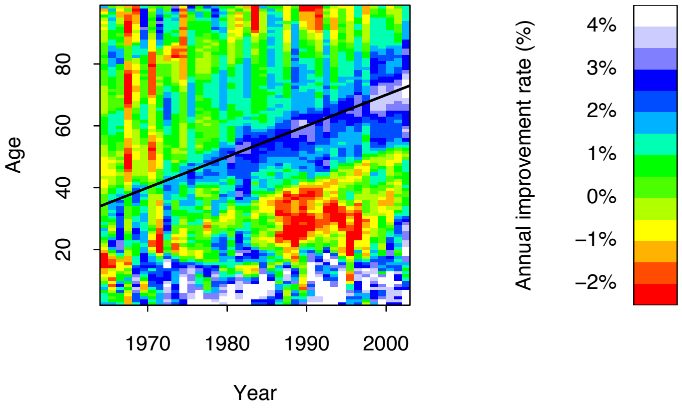
Cohort: group of individuals who have experienced the same event during the same period. (ex : birth cohort of individuals born in 1930)

$\Rightarrow$ Individuals of the same cohort will have similar demographic characteristics ("cohort effect").

Cohort effect, II

(Cairns *et al.*, 2009) $[r_{a,t} = (q_{a,t-1} - q_{a,t})/q_{a,t}]$

Golden cohort: generations born between 1925 and 1945



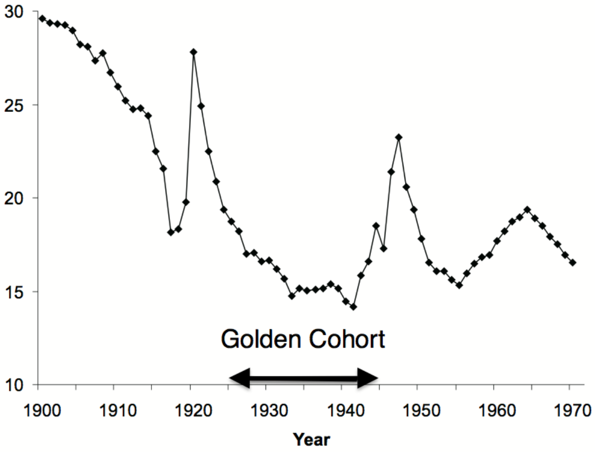
Cohort effect, III (UK)

The Cohort Effect: Insights And Explanations, 2004, R. C. Willets

The Golden cohort has experienced more rapid improvements than earlier and later generations. Some possible explanations:

- Impact of World War II on previous generations,
- Changes on smoking prevalence: tobacco consumption in next generations,
- Impact of diet in early life,
- Post World War II welfare state,
- Patterns of birth rates

Cohort effect, IV (UK)



Data source: www.mortality.org

Figure 6. Crude birth rate per 1,000 population, England and Wales, 1900 to 1970

Cohort effect, V (UK)

"One possible consequence of rapidly changing birth rates is that the 'average' child is likely to be different in periods where birth rates are very different. For instance, if trends in fertility vary by socio-economic class, the class mix of a population will change."

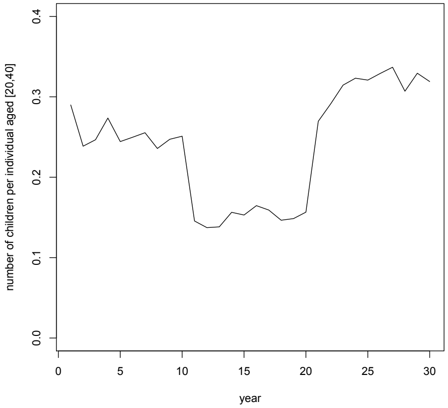
The Cohort Effect: Insights And Explanations, 2004, R. C. Willets

Simple toy model

- Reference death rate $\bar{d}(a) = A\exp(Ba)$
- Parameters $A \sim 0.0004$ and $B \sim 0.073$ estimated on French national data for year 1925 to capture a proper order of magnitude
- Group 1 : time independent death rate $d^1(a) = \bar{d}(a)$ and birth rate $b^1(a) = c\mathbf{1}_{[20,40]}(a)$ (c=0.1)
- Group 2 : time independent death rate $d^2(a) = 2\bar{d}(a)$ but birth rate $b^2(a,t) = 4c\mathbf{1}_{[20,40]}(a)\mathbf{1}_{[0,t_1]\cup[t_2,\infty)}(t)$
- Constant death rates but reduction in overall fertility between times t_1 (=10) and t_2 (=20)
- Aim: compute standard demographic indicators

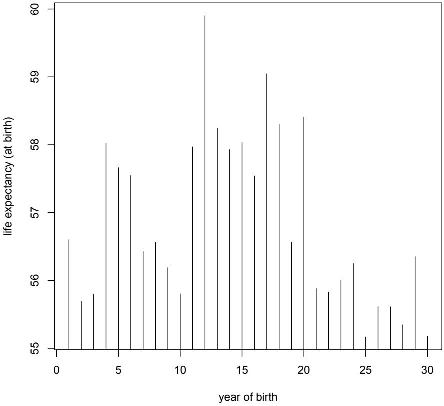
Aggregate fertility

- One trajectory with 10000 individuals (randomly) splitted between groups. Estimation of aggregate fertility



Life expectancy by year of birth

- "Cohort effect" for aggregate life expectancy



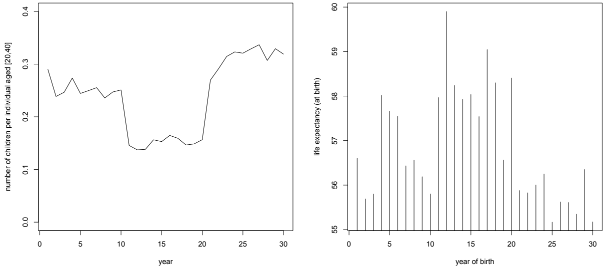


Figure : Observed fertility (left) and estimated life expectancy by year of birth (right)

- Death rates by specific group remain the same
- But reduction in fertility for "lower class" during 10-20 modifies the generations composition
  $\Rightarrow$ "upper class" is more represented among those born between 10 and 20

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