Polynomial approximation of mutivariate aggregate claim amounts distribution

Applications to reinsurance

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Sommaire

Multivariate risk models: Capturing dependency

Modelization of aggregate claim amounts associated to

- individuals in a portfolio,
- **Different lines of business** that belong to **a given insurance company**,
- **A given line of business** of that belong to **several insurance companies**.

Claims are caused by an event, and the severities are correlated to the **magnitude** of it.

- **Third-party liability motor**: Bodily injured claims and property damage claims
- **Workers' Compensation**: Medical claims and income replacement claims

Univariate risk model: How to include dependency?

A univariate total claim amount random variable is defined by,

$$X = \sum_{i=1}^{N} U_i,$$

- N is a counting random variable,
- $\{U_i\}_{i\in\mathbb{N}}$ is a sequence of **i.i.d.** random variables.

Dependency occurs

- Between the **claim severities**
- Between the **claim frequencies**

Model #1

Risk are modeled jointly via,

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ \vdots \\ V_{nj} \end{pmatrix},$$

- $\{\mathbf{V}_i\}_{i\in\mathbb{N}} = \{(V_{1i}, \dots, V_{ni})\}_{i\in\mathbb{N}}$ is a sequence of **i.i.d.** random vectors,
- M is a counting random variable,
- $\{\mathbf{V}_i\}_{i\in\mathbb{N}}$, and M are mutually independent.

Model #2

Risk are modeled jointly via,

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N_1} U_{1j} \\ \vdots \\ \sum_{j=1}^{N_n} U_{nj} \end{pmatrix},$$

- $\mathbf{N} = (N_1, \ldots, N_n)$ is a counting random vectors,
- $\{\mathbf{U}_i\}_{i\in\mathbb{N}} = (\{U_{1i}\}_{i\in\mathbb{N}}, \ldots, \{U_{ni}\}_{i\in\mathbb{N}})$ are independant sequences of **i.i.d.** random variables,
- **N**, and $\{\mathbf{U}_i\}_{i\in\mathbb{N}}$ are mutually independent.

Model #3

Risk are modeled jointly via,

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N_1} U_{1j} \\ \vdots \\ \sum_{j=1}^{N_n} U_{nj} \end{pmatrix} + \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ \vdots \\ V_{nj} \end{pmatrix},$$

- $\mathbf{N} = (N_1, \dots, N_n)$ is a counting random vectors,
- $\{\mathbf{U}_i\}_{i\in\mathbb{N}} = (\{U_{1i}\}_{i\in\mathbb{N}}, \dots, \{U_{ni}\}_{i\in\mathbb{N}})$ are independant sequences of **i.i.d.** random variables,
- $\{\mathbf{V}_i\}_{i\in\mathbb{N}} = \{(V_{1i}, \dots, V_{ni})\}_{i\in\mathbb{N}}$ is a sequence of **i.i.d.** random vectors,
- M is a counting random variable,
- $\{\mathbf{V}_i\}_{i\in\mathbb{N}}$, $\mathbf{N}$, M and $\{\mathbf{U}_i\}_{i\in\mathbb{N}}$ are mutually independent.

Two insurers and one reinsurer are in a pub.

Let 2 insurance portfolios associated to the same line of business that belong to two insurance companies,

- The aggregate claim amounts are given by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N_1} U_{1j} \\ \sum_{j=1}^{N_2} U_{2j} \end{pmatrix} + \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ V_{2j} \end{pmatrix},$$

- N_1 and N_2 are **independent**,
- Polynomial approximation of the joint **PDF** of (X_1, X_2)

A non-proportional global reinsurance treaty

Reinsurer offers insurer i a usual stop-loss reinsurance treaty with priority b_i and limit c_i, $i = 1, 2$.

The joint distribution of (X_1, X_2) is useful

- **Pricing purposes:** Determination of the stop loss premium.
- **Risk Management purposes:** The risk exposure of the reinsurer is modeled through,

$$Z = \min\left[(X_1 - b_1)_+ , c_1\right] + \min\left[(X_2 - b_2)_+ , c_2\right].$$

$\hookrightarrow$ *Value-at-Risk* computation and solvency capital determination.

A 2-dimensional Polynomial approximation

Let (X_1, X_2) be a random vector with probability measure $\mathbb{P}_{X_1,X_2}$, and **PDF** f_{X_1,X_2}.

- ν is a reference probability measure, constructed via the product of two univariate probability measure,

$$\begin{aligned} \nu(x_1, x_2) &= \nu_1(x_1) \times \nu_2(x_2) \\ f_\nu(x_1, x_2) &= f_{\nu_1}(x_1) \times f_{\nu_2}(x_2) \end{aligned}$$

- $\{Q_k^{\nu_i}\}_{k\in\mathbb{N}}$ is an orthonormal polynomial system ν_i, $i = 1, 2$.
- $\{Q_{k,l}\}_{k,l\in\mathbb{N}}$ is an orthonormal polynomial system ν, where

$$Q_{k,l}(x_1, x_2) = Q_k^{\nu_1}(x_1) Q_l^{\nu_2}(x_2), \quad k, l \in \mathbb{N}.$$

A 2-dimensional Polynomial approximation

If $\frac{d\mathbb{P}_{X_1,X_2}}{d\nu} \in L^2(\nu)$, then

$$\frac{d\mathbb{P}_{X_1,X_2}}{d\nu}(x_1, x_2) = \sum_{k,l=0}^{+\infty} a_{k,l} Q_{k,l}(x_1, x_2),$$

where

$$a_{k,l} = \mathbb{E}\left[Q_{k,l}(X_1, X_2)\right] = \mathbb{E}\left[Q_k^{\nu_1}(X_1) Q_l^{\nu_2}(X_2)\right]$$

The Parseval identity

$$\left\|\frac{d\mathbb{P}_{X_1,X_2}}{d\nu}\right\|^2 = \sum_{k,l=0}^{+\infty} a_{k,l}^2 < +\infty$$

is checked

Méthode d'approximation polynomiale en dimension 2

The PDF of (X_1, X_2) admits a polynomial representation

$$f_{X_1,X_2}(x_1, x_2) = \sum_{k,l=0}^{+\infty} a_{k,l} Q_{k,l}(x_1, x_2) f_\nu(x_1, x_2).$$

PDF Approximations follow from truncations,

$$f^{K,L}_{X_1,X_2}(x_1, x_2) = \sum_{k=0}^{K} \sum_{l=0}^{L} a_{k,l} Q_{k,l}(x_1, x_2) f_\nu(x_1, x_2),$$

where K, and L denote the orders of truncation. Survival function approximations follow from integration

$$\overline{F}^{K,L}_{X_1,X_2}(u_1, u_2) = \int_{u_1}^{+\infty} \int_{u_2}^{+\infty} \sum_{k=0}^{K} \sum_{l=0}^{L} a_{k,l} Q_{k,l}(x_1, x_2) f_\nu(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2.$$

The probability measure associated to

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N_1} U_{1j} \\ \sum_{j=1}^{N_2} U_{2j} \end{pmatrix} + \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ V_{2j} \end{pmatrix}$$
$$= \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} + \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix},$$

has a lot of singularities.

On the choice of the reference probability measure

- The probability measure of $\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ V_{2j} \end{pmatrix}$, is

$$d\mathbb{P}_{Y_1,Y_2}(y_1,y_2) = f_M(0)\delta_{0,0}(y_1,y_2) + d\mathbb{G}_{Y_1,Y_2}(y_1,y_2).$$

$d\mathbb{G}_{Y_1,Y_2}$ is a defective probability measure having support on $\mathbb{R}_+^2$.

- ν is defined as the product of gamma measures.
  - $\hookrightarrow$ ν_i is a gamma measure $\Gamma(m_i, r_i)$, $i = 1, 2$.

$$f_{\nu_i}(x) = \frac{e^{-x/m_i} x^{r_i}}{\Gamma(r_i) m_i^{r_i}}, \qquad x \geq 0.$$

  - $\hookrightarrow$ $\{Q_k^{\nu_i}\}_{k\in\mathbb{N}}$ is sequence of generalized Laguerre polynomials, $i = 1, 2$.

On the choice of the reference probability measure

- The probability measure of $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N_1} U_{1j} \\ \sum_{j=1}^{N_2} U_{2j} \end{pmatrix}$, is

$$
\begin{aligned}
d\mathbb{P}_{W_1,W_2}(w_1, w_2) &= f_{N_1}(0)f_{N_2}(0)\delta_{0,0}(w_1, w_2) \\
&+ d\mathbb{G}_{W_1}(w_1) \times d\mathbb{G}_{W_2}(w_2) \\
&+ f_{N_1}(0)d\mathbb{G}_{W_2}(w_2) \times \delta_0(w_1) \\
&+ f_{N_2}(0)d\mathbb{G}_{W_1}(w_1) \times \delta_0(w_2).
\end{aligned}
$$

- Univariate polynomial approximation for $\mathbb{G}_{W_i}$, $i = 1, 2$.
  $\hookrightarrow$ Gamma probability measure and generalized Laguerre polynomials.

On the integrability condition

Theorem

Let $\mathbf{Y} = (Y_1, Y_2)$ be a random vector. If

H1 The set

$$\mathbf{\Gamma_Y} = \inf\{(s_1, s_2) \in \mathbb{R}^2_{+*}, \mathcal{L}_{\mathbf{Y}}(s_1, s_2) = +\infty\},$$

is not empty.

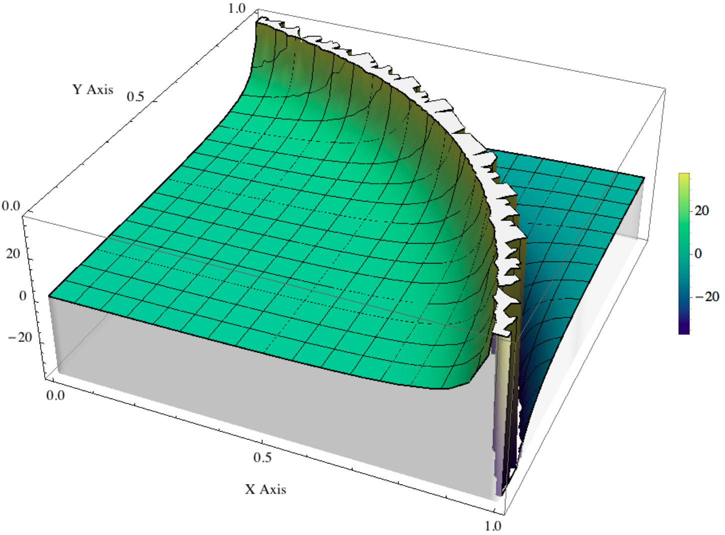
H2 There exists $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2_+$ such that $y_i \mapsto f_{\mathbf{Y}}(y_1, y_2)$ are strictly decreasing for $y_i \geq a_i$, and $i = 1, 2$.

Then for $\mathbf{y} \geq \mathbf{a}$,

$$f_{\mathbf{Y}}(y_1, y_2) \leq A_{\mathbf{Y}}(s_1, s_2) e^{-(s_1 y_1 + s_2 y_2)}, \quad \mathbf{s} \leq \gamma_Y,$$

where $\gamma_Y \in \mathbf{\Gamma_Y}$.

On the integrability condition



On the integrability condition

1. Find $(\gamma_{Y_1}, \gamma_{Y_2})$ such that

$$\mathcal{L}_{Y_1,Y_2}(\gamma_{Y_1}, \gamma_{Y_2}) < +\infty$$

   and

$$\gamma_{Y_1}, \gamma_{Y_2} > 0$$

2. Choose r_1, r_2, m_1, and m_2 such that

$$0 < r_1, r_2 \leq 1$$

   and

$$0 < \frac{1}{m_1} < 2\gamma_{Y_1}, \qquad 0 < \frac{1}{m_2} < 2\gamma_{Y_2}.$$

Decay of $\{a_{k,l}\}_{k,l\in\mathbb{N}}$: Generating function study

The defective **PDF** of (Y_1, Y_2) admits the polynomial representation

$$g_{Y_1,Y_2}(y_1,y_2) = \sum_{k,l=0}^{+\infty} a_{k,l} Q_{k,l}(y_1,y_2) f_{\nu}(y_1,y_2).$$

Taking the Laplace transform leads to

$$\mathcal{C}(z_1,z_2) = (1+z_1)^{-r_1}(1+z_2)^{-r_2} L_{g_{Y_1,Y_2}}\left[\frac{z_1}{m_1(1+z_1)}, \frac{z_2}{m_2(1+z_2)}\right],$$

where $\mathcal{C}(z_1,z_2) = \sum_{k,l=0}^{+\infty} a_{k,l} c_k^{\nu_1} c_l^{\nu_2} z_1^k z_2^l$, and

$$c_k^{\nu_i} = \sqrt{\binom{k+r_i-1}{k}}, \quad i = 1,2.$$

Numerical illustrations: Survival function of (Y_1, Y_2)

- M is governed by a geometric distribution $\mathcal{NB}(1, 3/4)$
- (V_1, V_2) is governed by a bivariate exponential distribution $DBVE(\rho, \mu_1, \mu_2)$
  - ↪ $\rho = \frac{1}{4}$
  - ↪ $\mu_1 = \mu_2 = 1$
- Polynomial approximations are compared to Monte-Carlo based approximations.

The parametrization

$$m_1 = \frac{1}{(1-p)\mu_1},\quad m_2 = \frac{1}{(1-p)\mu_2},\quad r_1 = r_2 = 1.$$

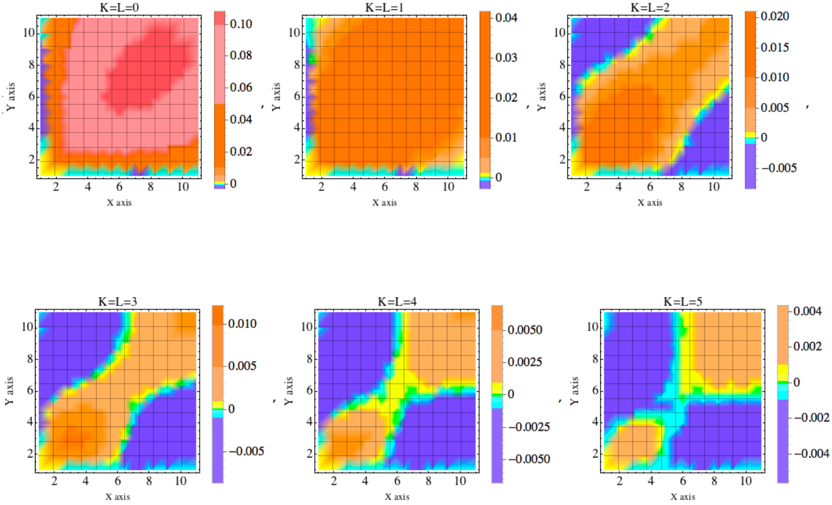
leads to a generating function of the form

$$\mathcal{C}(z_1, z_2) = \frac{1}{1 + z_1 z_2 (p^2 - \rho(1-p)^2 - p)}.$$

and

$$a_{k,l} = \left[p^2 - \rho(1-p)^2 - p\right]^k \delta_{kl},\quad k, l \in \mathbb{N}$$

Numerical illustrations:
Survival function of (Y_1, Y_2)



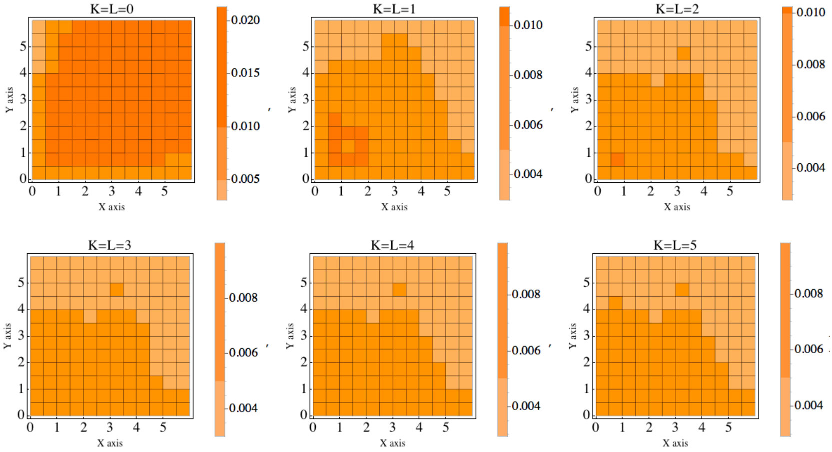
Numerical illustrations: Distribution of (X_1, X_2)

- N_1 and N_2 are geometrically distributed $\mathcal{NB}(1, 3/4)$,
- $\{U_{1j}\}_{j\in\mathbb{N}}$ are $\{U_{2j}\}_{j\in\mathbb{N}}$ are sequences of **i.i.d.** $\Gamma(1,1)$-distributed,
  $\hookrightarrow$ The **PDF** is available in a closed form.
- Polynomial approximation of the survival function of (X_1, X_2) with increasing truncation order.
- Priorities: $c_1 = c_2 = 1$.
- Limits: $b_1 = b_2 = 4$.
- Polynomial approximations with orders of truncation equal to 10 of the survival function of

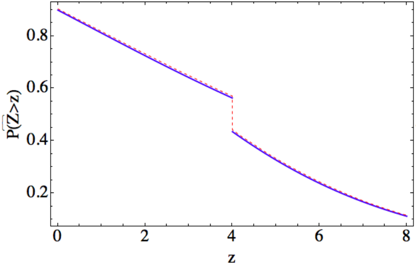
$$Z = \min\left[(X_1 - b_1)_+, c_1\right] + \min\left[(X_2 - b_2)_+, c_2\right].$$

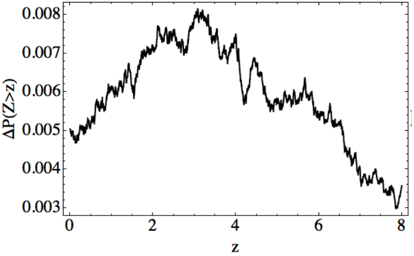
- The polynomial approximations are compared to Monte Carlo approximations.

Numerical illustrations: Survival function of (X_1, X_2)



Numerical illustrations: Reinsurance cost





z	Monte Carlo approximation	Polynomial approximation
0	0.90385	0.898808
2	0.73193	0.724774
4	0.44237	0.435013
6	0.24296	0.237576

Conclusion and Perspectives

- The polynomial approximation is an efficient numerical method:
  - ↪ A good approximation of a bivariate aggregate claim amounts going along with motivation in reinsurance.

Perspectives

- Applications of the method to the approximation of other function in actuarial science and other fields of applied probabilities.
- Statistical extensions when data are available,

↪ The approximation can turn into a semi-parametric estimator of the **PDF**.