Orthonormal polynomial expansions and lognormal sum densities

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Goal: Recover the PDF of the sum of n lognormally distributed random variables.

Dependence

The lognormals in the sum may be correlated.

Extreme Value

The lognormal distribution is heavy tailed.

Actuarial Science

Modelization of the total claim amounts of non life insurance portfolios.

Further aplications...

Finance

Black & Scholes model $\Rightarrow$ security prices are lognormally distributed,

- The value of the portfolio is distributed as a sum of lognormally distributed random variable,
- Pricing of Asian option.
  - $\hookrightarrow$ The pay off is determined by the average of the underlying price.

Telecommunication

The inverse of the signal to noise ratio can be modeled as a sum of **i.i.d.** lognormals.

Definition

The random variable defined as

$$S = e^{X_1} + \ldots + e^{X_n}$$

where $(X_1, \ldots, X_n) \sim \mathcal{MN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ admits a sum of lognormals distribution $\mathcal{SLN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

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$$f_S(x) = \text{ ???}$$

The PDF is unknown even in the case of the sum of two **i.i.d.** lognormals.

Let X be a random variable governed by $\mathbb{P}_X$, with unknown PDF f_X.

I. Pick a reference probability measure ν

- $\mathbb{P}_X$ absolutely continuous **w.r.t.** ν.
- f_ν as first approximation of f_X.

II. $\{Q_k\}_{k\in\mathbb{N}}$ sequence of orthonormal polynomials **w.r.t.** ν

- Gram-Schmidt orthogonalization procedure.

III. Orthogonal projection of f_X/f_ν onto $\{Q_k\}$

- Adjustment of the first approximation.

$\{Q_k\}_{k\in\mathbb{N}}$ must be complete in a set of functions in which f_X/f_ν belongs.

The $\mathcal{L}^2(\nu)$ space

- The set of square integrable functions with respect to ν.
- Inner product,

$$\langle f, g\rangle = \int f(x)g(x)\mathrm{d}\nu(x), \text{ where } f, g \in \mathcal{L}^2(\nu).$$

- $\{Q_k\}_{k\in\mathbb{N}}$ is a sequence of orthogonal polynomials **w.r.t.** ν in the sense that

$$\langle Q_k, Q_l\rangle = \int Q_k(x)Q_l(x)\mathrm{d}\nu(x) = \delta_{kl}.$$

Sufficient Condition

If

$$\int e^{\alpha|x|}\mathrm{d}\nu(x) < +\infty,$$

then $\{Q_k\}_{k\in\mathbb{N}}$ is complete in $\mathcal{L}^2(\nu)$.

Orthonormal Polynomial Expansion

If $f_X/f_\nu \in \mathcal{L}^2(\nu)$ then

$$f_X(x) = \sum_{k=0}^{+\infty} a_k Q_k(x) f_\nu(x),$$

where

$$a_k = \left\langle Q_k, \frac{f_X}{f_\nu} \right\rangle = \int Q_k(x)\frac{f_X(x)}{f_\nu(x)}\mathrm{d}\nu(x) = \mathbb{E}\left[Q_k(X)\right].$$

Orthonormal polynomial approximation

The approximation follows from simple truncation

$$f_X^K(x) = \sum_{k=0}^{K} a_k Q_k(x) f_\nu(x).$$

The accuracy relies on the decay of the a_k's.

Lognormal distribution as reference

$$f_\nu(x) = \frac{1}{x\sigma\sqrt{2\pi}} \mathrm{e}^{-\left(\frac{\ln(x)-\mu}{\sigma\sqrt{2}}\right)^2}, \qquad x > 0.$$

Associated Orthogonal Polynomials

$$Q_k(x) = \mathrm{e}^{-\frac{k^2\sigma^2}{2}} \sum_{i=0}^{k} \frac{(-1)^{k+i}\mathrm{e}^{-i\mu-\frac{i^2\sigma^2}{2}}}{\sqrt{\left[\mathrm{e}^{-\sigma^2}, \mathrm{e}^{-\sigma^2}\right]_k}} e_{k-i}\left(1, \ldots, \mathrm{e}^{(k-1)\sigma^2}\right) x^i,$$

for $k \in \mathbb{N}$ where $e_i\left(X_1, \ldots, X_k\right)$ are the elementary symmetric polynomials and $[x, q]_n = \prod_{i=0}^{n-1}\left(1 - xq^i\right)$ is the Q-Pochhammer symbol.

Let S be $\mathcal{SLN}(\boldsymbol{\mu}, \boldsymbol{\sigma})$-distributed.

Integrability Condition

If

$$f_S(x) = \mathcal{O}\left(\mathrm{e}^{-b\log^2 x}\right) \quad \text{for } x \to 0 \text{ and } \infty,$$

where $b > \left(4\sigma^2\right)^{-1}$, then $f_S/f_\nu \in \mathcal{L}^2(\nu)$

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Asymptotics for f_S

$$f_S(x) = \mathcal{O}(\exp\{-c_1 \ln(x)^2\}) \text{ as } x \to 0,$$

where $c_1 = \left[2 \min_{\mathbf{w}\in\Delta} \mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}\mathbf{w}\right]^{-1}$, see Tankov et al [GT15].
and

$$f_S(x) = \mathcal{O}(\exp\{-c_2 \ln(x)^2\}) \text{ as } x \to \infty$$

where $c_2 = \left[2 \max_{i=1,\ldots,n} \Sigma_{ii}\right]^{-1}$, see Asmussen et al. [ARN08]

The integrability condition is satisfied as long as

$$\sigma^2 > \frac{\max \Sigma_{ii}}{2}.$$

PB1 Is $\{Q_k\}_{k\in\mathbb{N}}$ complete in $\mathcal{L}^2(\nu)$?

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Proposition

$\Rightarrow$ No, it's not!

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PB1 Is $\{Q_k\}_{k\in\mathbb{N}}$ complete in $\mathcal{L}^2(\nu)$?

Proposition

$\Rightarrow$ No, it's not!

PB2 The lognormal distribution is not characterized by its moments and neither is the sum of lognormals distribution.

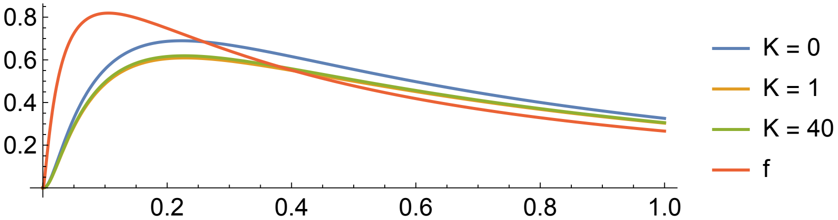


Fig. 1 orthogonal polynomial approximations of $\mathcal{LN}(0, 1.50^2)$ using a $\mathcal{LN}(0, 1.22^2)$ reference.

Normal distribution $\mathcal{N}(\mu, \sigma)$ as reference

$$f_\nu(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2}, \qquad x \in \mathbb{R}.$$

Associated Orthonormal Polynomials

$$Q_k(x) = \frac{1}{k!2^{k/2}} H_k\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where $\{H_k\}_{k\in\mathbb{N}}$ are the Hermite polynomials, see Szego [Sze39].

- $f_S/f_\nu \notin \mathcal{L}^2(\nu)$.
  $\hookrightarrow$ We consider a transformation of S

$$Z = \ln(S).$$

Integrability Condition

$$f_Z(x) = \mathcal{O}\left(\mathrm{e}^{-ax^2}\right) \quad \text{as } x \to \pm\infty,$$

where $a > \left(4\sigma^2\right)^{-1}$.

Asymptotics for f_Z

$$f_Z(z) = \mathcal{O}(\exp\{-c_1 z^2\}) \text{ as } z \to -\infty$$

and

$$f_Z(z) = \mathcal{O}(\exp\{-c_2 z^2\}) \text{ as } z \to +\infty$$

Extension of the work of Gao et al. [GXY09].

The integrability condition is satisfied as long as

$$\sigma^2 > \frac{\max \Sigma_{ii}}{2}.$$

- f_Z is approximated by

$$\widehat{f}_Z(z) = \sum_{k=0}^{K} a_k Q_k(z) f_\nu(z),$$

where the coefficient $a_k = \mathbb{E}\{Q_k[\ln(S)]\}$ are evaluated using Crude Monte Carlo for $k = 1, \ldots, K$.

- f_S is approximated by

$$\widehat{f}_{S,\mathcal{N}}(x) = \frac{1}{x}\widehat{f}_Z(\ln(x)).$$

Gamma Distribution $\Gamma(m, r)$ As Reference

$$f_{\nu}(x) = \frac{\mathrm{e}^{-x/m} x^{r-1}}{\Gamma(r) m^r}, \ x > 0.$$

Associated Orthonormal Polynomials

$$Q_n(x) = (-1)^n \left[\frac{\Gamma(n+r)}{\Gamma(n+1)\Gamma(r)} \right]^{-1/2} L_n^{r-1}(x/m),$$

where the $\{L_n^{r-1}\}_{k\in\mathbb{N}}$ are the generalized Laguerre polynomials defined in Szego [Sze39].

- $f_S/f_\nu \notin \mathcal{L}^2(\nu)$.
  - $\hookrightarrow$ We consider the PDF of the exponentially tilted distribution of S
$$f_\theta(x) = \frac{\mathrm{e}^{-\theta x} f_S(x)}{\mathcal{L}(\theta)},$$
where $\mathcal{L}(\theta) = \mathbb{E}\left(\mathrm{e}^{-\theta S}\right)$.

Integrability Condition

If
$$f_\theta(x) = \mathcal{O}\left(\mathrm{e}^{-\delta x}\right) \quad \text{for } x \to +\infty,$$
where $\delta > \frac{1}{2m}$, and
$$f_\theta(x) = \mathcal{O}\left(x^{\beta}\right) \quad \text{for } x \to 0,$$
where $\beta > r/2 - 1$, then $f_\theta/f_\nu \in \mathcal{L}^2(\nu)$.

The integrability condition is satisfied as long as

$$m > \frac{1}{2\theta}.$$

- f_θ is approximated by

$$\widehat{f_\theta}(x) = \sum_{k=0}^{K} a_k Q_k(x) f_\nu(x),$$

where the coefficient $a_k = \mathbb{E}\left[Q_k(S_\theta)\right]$ with $S_\theta \sim f_\theta$. f_S is approximated by

$$\widehat{f_{S,\Gamma}}(x) = \mathcal{L}(\theta)\mathrm{e}^{\theta x}\widehat{f_\theta}\left(x\right).$$

Regarding the evaluation of the a_k's, we have

$$a_k = \mathbb{E}\left[Q_k(S_\theta)\right] = q_{k0} + q_{k1}\mathbb{E}(S_\theta) + \ldots + q_{kk}\mathbb{E}(S_\theta^k)$$

where the q_{ki}'s are the coefficients of Q_k, and

$$\mathbb{E}(S_\theta^i) = \frac{\mathbb{E}\left(S^i e^{-\theta S}\right)}{\mathcal{L}(\theta)} = \frac{\mathcal{L}_i(\theta)}{\mathcal{L}(\theta)}$$

where $\mathcal{L}_i(\theta)$ denotes the i^{th} derivative of $\mathcal{L}(\theta)$.

- We adapt techniques developed in Asmussen et al. [AJRN14, LAJRN16] to access $\mathcal{L}_i(\theta)$.

Polynomial Aproximation Settings

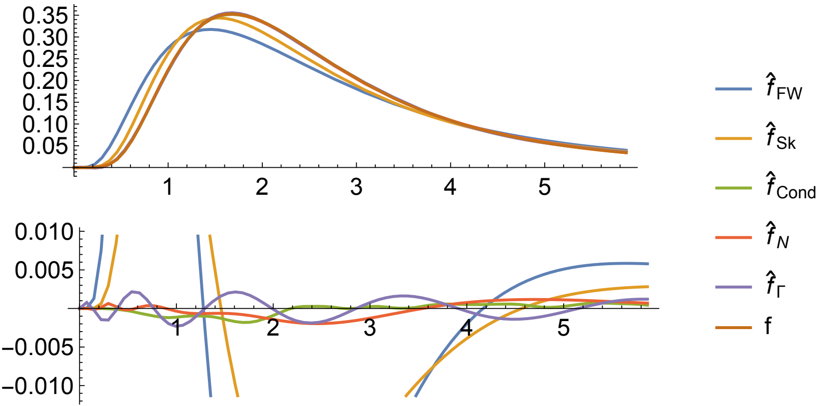
- Normal distribution as reference
  - $\mu = \mathbb{E}(Z)$ and $\sigma^2 = \mathbb{V}(Z)$
  - 10^5 replications for the CMC procedure
- Gamma distribution as reference
  - $\theta = 1$
  - Moment matching of order 2 between f_θ and f_ν to get m and r.

Some Challengers

- The Fenton-Wilkinson approximation $\widehat{f}_{FW}$, see [Fen60]
- The log skew normal approximation $\widehat{f}_{SK}$, see [HB15]
- The conditionnal Monte Carlo approximation $\widehat{f}_{Cond}$, cf. Example 4.3 on p. 146 of [AG07].

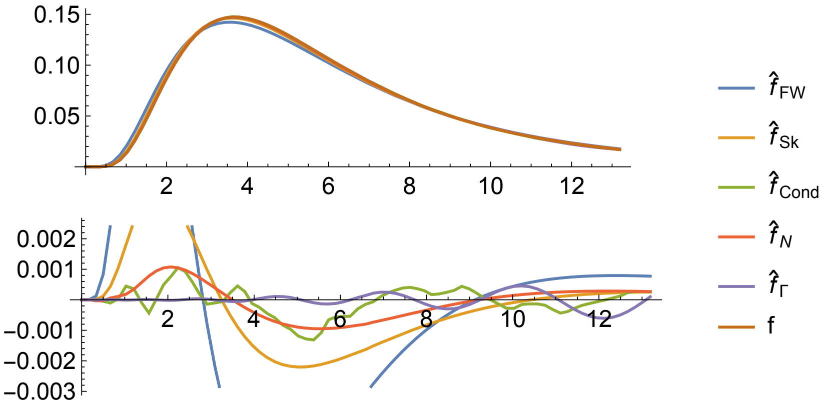
A benchmark is obtained through numerical integration.

- $f_S(x)$ and $\widehat{f}_S(x)$ together with $\left(\widehat{f}_S(x) - f_S(x)\right)$ are plotted over $x \in [0, 2\mathbb{E}(S)]$,
- The $\mathcal{L}^2$ error are computed over $[0, \mathbb{E}(S)]$.



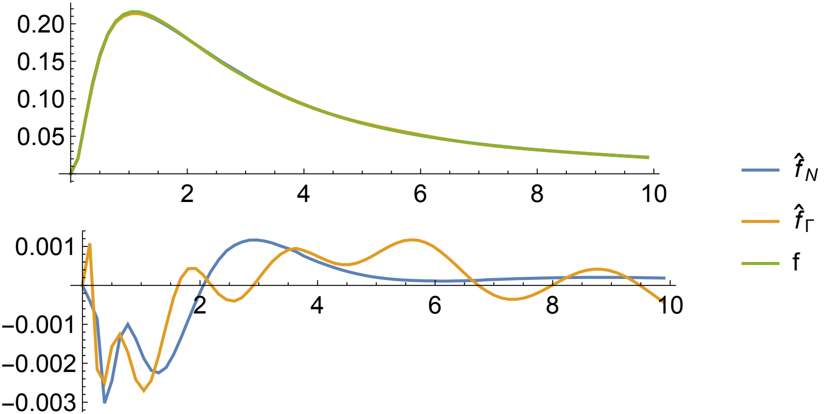
$\widehat{f_{\mathrm{FW}}}$	$\widehat{f_{\mathrm{Sk}}}$	$\widehat{f_{\mathrm{Cond}}}$	$\widehat{f_{\mathcal{N}}}$	$\widehat{f_{\Gamma}}$
8.01×10^{-2}	4.00×10^{-2}	1.56×10^{-3}	1.94×10^{-3}	2.28×10^{-3}

Test 1 ***mu*** $= (0,0)$, $\mathrm{diag}(\boldsymbol{\Sigma}) = (0.5, 1)$, $\rho = -0.2$. Reference distributions used are $\mathcal{N}(0.88, 0.71^2)$ and Gamma$(2.43, 0.51)$ with $K = 32$.



$\widehat{f_{FW}}$	$\widehat{f_{Sk}}$	$\widehat{f_{Cond}}$	$\widehat{f_{\mathcal{N}}}$	$\widehat{f_{\Gamma}}$
1.82×10^{-2}	6.60×10^{-3}	1.90×10^{-3}	1.80×10^{-3}	1.77×10^{-4}

Test 2 $n = 4$, $\mu_i = 0$, $\Sigma_{ii} = 1$, $\rho = 0.1$. Reference distributions used are $\mathcal{N}(1.32, 0.74^2)$ and Gamma$(3.37, 0.51)$ with $K = 32$



Test 6 Sum of 3 $\mathcal{LN}(0,1)$ r.v.s with $C_{10}^{\text{Cl}}(\cdot)$ copula (i.e., $\tau = \frac{5}{6}$). Reference distributions used are $\mathcal{N}(1.46, 0.71^2)$ and Gamma$(8.78, 0.25)$ with $K = 40$. The $\mathcal{L}^2$ errors of $\widehat{f}_{\mathcal{N}}$ and $\widehat{f}_\Gamma$ are 2.45×10^{-3} and 2.04×10^{-3} respectively.

- The orthonormal polynomial approximation is an efficicient numerical method: Accurate and easy to code.
- None of the method tested here seem to universally superior to the other.
- Good behavior in presence of a non-Gaussian dependence structure: cf. test 3 with Clayton copula.
- For more details, the paper is available on ArXiv [AGL16].

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The symmetric polynomials are defined as

$$e_i\left(X_1,\ldots,X_k\right)=\begin{cases}\sum_{1\leq j_1<\ldots<j_i\leq k} X_{j_1}\ldots X_{j_i}, & \text{for } i\leq k,\\ 0, & \text{for } i>k,\end{cases}$$

and $[x,q]_n=\prod_{i=0}^{n-1}\left(1-xq^i\right)$ is the Q-Pochhammer symbol.

Log Skew Normal Approximation

X is governed by a skew normal distribution $\mathcal{SN}(\lambda, \omega, \epsilon)$ if its PDF is given by

$$f_X(x) = \frac{2}{\omega}\phi\left(\frac{x-\epsilon}{\omega}\right)\Phi\left(\lambda\frac{x-\epsilon}{\omega}\right),$$

where ϕ and Φ are respectively the PDF and CDF of the standard normal distribution.

$\Rightarrow$ $Y = e^X$ is governed by a log skew normal distribution $\mathcal{LSN}(\lambda, \omega, \epsilon)$

- Central moment matching (the two first one) with the sum of lognormals distribution.
- Left tail slope matching using the asymptotics derived in [GT15].

Conditional Monte Carlo Approximation

By definition, the conditional Monte Carlo is the Crude Monte Carlo estimator of the representation

$$f_S(x) = \mathbb{E}\left\{\mathbb{P}(S \in \mathrm{d}x | Y)\right\}$$

Recall that $S = \mathrm{e}^{X_1} + \ldots + \mathrm{e}^{X_n}$, and set $Y = e^{X_1}$, then the conditional Monte Carlo estimator is written as follows

$$\widehat{f}_{\mathrm{Cond}} = \mathbb{E}\left\{ f_{\mathrm{e}^{X_1}}\left[x - (S - e^{X_1})\right] \Big| X_1 \right\},$$

where $f_{e^{X_1}}$ is the PDF of a lognormal distribution $\mathcal{LN}\left[\mathbb{E}(X_1), \mathbb{V}(X_1)\right]$

Approximation of $\mathcal{L}(\theta)$

Precisions regarding the evaluation of the a_k's in the gamma case

The Laplace transform of S is given by

$$\mathcal{L}(\theta) = \mathbb{E}\left(\mathrm{e}^{-\theta S}\right) \propto \int \exp\left(-h_\theta(\mathbf{x})\right) \mathrm{d}\mathbf{x}$$

where

$$h_\theta(\mathbf{x}) = \theta(\mathrm{e}^{\mu})^T \mathrm{e}^{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T \mathbf{D}\mathbf{x},$$

where $\mathbf{D} = \mathbf{\Sigma}^{-1}$.

- Replace h_θ by its second order Taylor expansion arround its unique minimizer $\mathbf{x}^*$

$$-\left(1-\frac{1}{2}\mathbf{x}^*\right)^T \boldsymbol{D}x^* + \frac{1}{2}(\mathbf{x}-\mathbf{x}^*)^T(\boldsymbol{\Lambda}+\mathbf{D})(\mathbf{x}-\mathbf{x}^*),$$

where $\boldsymbol{\Lambda} = \theta \mathrm{diag}\left(e^{\mu+\mathbf{x}^*}\right)$, to get the approximation $\tilde{\mathcal{L}}(\theta)$.

- The reinjection of the Taylor expansion in the initial Laplace transform expansion permits to get a correction term

$$\mathcal{L}(\theta) = \tilde{\mathcal{L}}(\theta) I(\theta),$$

where

$$I(\theta) = \sqrt{\det(\mathbf{I}+\boldsymbol{\Sigma}\boldsymbol{\Lambda})}\,\mathbb{E}\left[v\left(\Sigma^{1/2}Z\right)\right],$$

with $v(\mathbf{u}) = \exp\left\{(\mathbf{x}^*)^T\mathbf{D}(e^{\mathbf{u}}-\mathbf{1}-\mathbf{u}\right\}$ and $Z \sim \mathcal{N}(0,1)$.

More detail can be found in [LAJRN16]