A polynomial expansion to approximate ruin probabilities

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Sommaire

Executive summary

Main goal

Work out a new numerical method to approximate ruin probabilities.

main idea

Polynomial expansion of a probability density function though orthogonal projection

- ↪ Change of measure via Natural Exponential Family with Quadratic Variance function
- ↪ Construction of a polynomial orthogonal system w.r.t this probability measure

Achievement

Approximation of the ultimate ruin probability in the compound Poisson ruin model with light tailed claim sizes

Notations

dF is an univariate Probability Measure. Denote by

- F its Cumulated Distribution Function,
- $f = F'$ its Probability density Function w.r.t. a positive measure,
- $\widehat{F}(\theta) = \int e^{\theta x} dF(x)$ its Moment Generating Function,
- $\kappa(\theta) = ln(\widehat{F}(\theta))$ its Cumulant Generating Function,

$L^2(F)$ is a function space such that :

- If $f \in L^2(F)$ then $\int f^2(x) dF(x) < \infty$.

$L^2(F)$ is a normed vector space :

$$||f||^2 = < f, f > = \int f^2(x) dF(x).$$

Definition and hypothesis

Denote by $\{R(t); t \geq 0\}$ the Risk Reserve Process :

$$R(t) = u + pt - \sum_{i=1}^{N(t)} U_i,$$

where

- u is teh initial reserves,
- p is the constant premium rate per unit of time,
- $N(t)$ is an homogeneous Poisson process with intensity β,
- $\{U_i\}_{i\in\mathbb{N}^*}$ are **i.i.d.** non-negative random variables, independant of $N(t)$, with CDF F_U and mean μ.

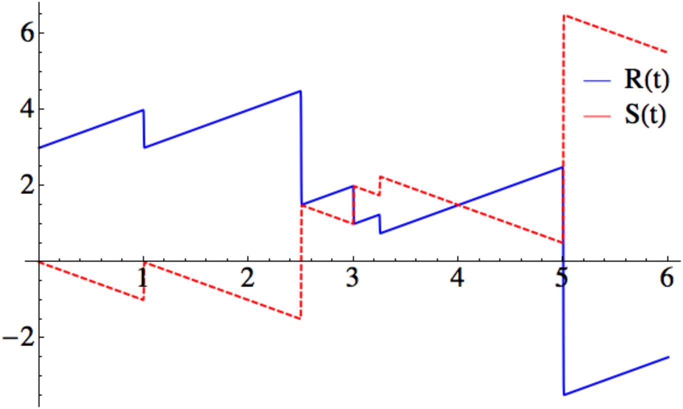
Let $\{S(t); t \geq 0\}$ be the Surplus process :

$$S(t) = u - R(t).$$

$\eta > 0$ is the safety loading and one had better make sure that :

$$p = (1+\eta)\beta\mu.$$

Risk and surplus processes visualization



Ultimate ruin probability

Denote by $M = Sup\{S(t); t > 0\}$, the ultimate ruin probability is defined as :

$$\psi(u) = P(M > u) = \overline{F_M}(u).$$

Pollaczek-Khinchine formula

In the compound Poisson ruin model, the ruin probability can be written as :

$$\psi(u) = (1-\rho)\sum_{n=0}^{+\infty}\rho^n\overline{F_{U^I}^{*n}}(u),$$

$$M \stackrel{D}{=} \sum_{i=1}^{N} U_i^I, \qquad F_{U^I}(x) = \int_0^x \frac{\overline{F_U(y)}}{\mu}dy,$$

where N is geometric with parameter $\rho = \frac{\beta\mu}{p} < 1$ and $F_{U^I}^{*n}$ denotes the nth convolution of F_{U^I}.

See *Ruin probabilities* par Asmussen et Albrecher (2001) [1].

Numerical evaluation of ruin probability : a brief review

- Panjer's algorithm, Panjer (1981) [8]
- Laplace transform numerical inversion
  - → Fast Fourrier Transform, Embrecht et al. (1993) [5]
  - → Laguerre's method, Weeks (1966) [10]
- Weighted sum of Gamma densities, Bowers (1966) [4]
  - → Beekman-Bowers Approximation, Beekman (1969) [3]
- Monte-Carlo simulations method, Kaasik (2009) [6]

Natural Exponential Families with Quadratic Variance Function

Let dF be a univariate probability measure possessing MGF in a Neighborhood of 0.

- $\{F_\mu; \mu \in \mathcal{M}\}$ is the NEF generated by dF, with :

$$dF_\mu(x) = exp(\phi(\mu)x - \kappa(\phi(\mu)))dF(x).$$

The variance function is said quadratic if :

$$V(\mu) = a\mu^2 + b\mu + c$$

The NEF-QVF contain six distribution :

- Normal
- Gamma
- Hyperbolic
- Binomial
- Negative Binomial
- Poisson

Orthogonal polynomials for NEF-QVF Distributions

Define $\{F_\mu; \mu \in M\}$ a NEF-QVF generated by dF with mean μ_0.

- The PDF $f(x,\mu)$, of a F_μ w.r.t. dF is proportional to $exp(\phi(\mu)x - \kappa(\phi(\mu)))$.

$$Q_n(x,\mu) = V^n(\mu)\left\{\frac{\partial^n}{\partial\mu^n}f(x,\mu)\right\}/f(x,\mu),$$

  is a polynomial of degree n in both μ and x.

- $f(x,\mu_0) = 1$ et

$$Q_n(x) = Q_n(x,\mu_0) = V^n(\mu_0)\left\{\frac{\partial^n}{\partial\mu^n}f(x,\mu)\right\}_{\mu=\mu_0}.$$

$\{Q_n\}$ is an orthogonal polynomials system such that :

$$< Q_n(x), Q_m(x) >= \int Q_n(x)Q_m(x)dF(x) = ||Q_n||^2\delta_{nm}.$$

For a full description of the NEF-QVF see Barndorf-Nielsen (1978) [2] et Morris (1982) [7].

Polynomial Expansion and Truncations

- The polynomials are dense in $L^2(F)$.
  $\hookrightarrow$ $\{Q_n\}$ is therefore an orthogonal basis of $L^2(F)$.
- Let dF_X be a probability measure associated to some random variable X.
  $\hookrightarrow$ $\frac{dF_X}{dF}$ its density w.r.t. dF
- If $\frac{dF_X}{dF} \in L^2(F)$ we have :

$$\frac{dF_X}{dF}(x) = \sum_{n\in\mathbb{N}} < \frac{dF_X}{dF}, \frac{Q_n}{||Q_n||} > \frac{Q_n(x)}{||Q_n||} = \sum_{n\in\mathbb{N}} E(Q_n(X))\frac{Q_n(x)}{||Q_n||^2}.$$

- The CDF F_X is then :

$$F_X(x) = \sum_{n\in\mathbb{N}} E(Q_n(X))\frac{\int_{-\infty}^{x} Q_n(y)dF(y)}{||Q_n||^2}.$$

Approximations are then obtained by truncation

$$F_X^K(x) = \sum_{n=0}^{K} E(Q_n(X))\frac{\int_{-\infty}^{x} Q_n(y)dF(y)}{||Q_n||^2}.$$

Polynomial expansion for the ultimate ruin probability

Recall that $M = \sum_{i=1}^{N} U_i^I$ then :

$$
\begin{aligned}
dF_M(x) &= (1-\rho)\delta_0(dx) + (1-\rho)\sum_{n=1}^{+\infty}\rho^n dF_{U^I}^{*n}(x) \\
&= (1-\rho)\delta_0(dx) + dG_M(x).
\end{aligned}
$$

If $\frac{dG_M}{dF} \in L^2(F)$ then :

$$
\frac{dG_M}{dF}(x) = \sum_{n\in\mathbb{N}} < \frac{dG_M}{dF}, \frac{Q_n}{||Q_n||} > \frac{Q_n(x)}{||Q_n||}.
$$

Integration leads to the polynomial expansion of the ruin probability :

$$
\psi(u) = \sum_{n\in\mathbb{N}} < \frac{dG_M}{dF}, \frac{Q_n}{||Q_n||} > \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n||}.
$$

Approximation of the ruin probability though truncation of the polynomial expansion

Approximation of the ultimate ruin probability

- $\{F_\mu; \mu \in M\}$ is a NEF-QVF generated by F with μ_0,
- $f(x,\mu) \propto exp(\phi(\mu)x - \kappa(\phi(\mu)))$ is the PDF of F_μ w.r.t. F.

If $\frac{dG_M}{dF} \in L^2(dF)$ then :

$$\begin{aligned}\psi^K(u) &= \sum_{n=0}^{K} V_F(\mu_0)^n \left[\frac{\partial^n}{\partial\mu^n} e^{-\kappa(\phi(\mu))}\left(\widehat{G_M}(\phi(\mu))\right)\right]_{\mu=\mu_0} \\ &\times \frac{\int_u^{+\infty} Q_n(y)dF(y)}{||Q_n(x)||^2}\end{aligned}$$

Choice of the NEF-QVF

dG_M is a defective probability measure supported on $[0,+\infty[$.
Among NEF-QVF, the only one suported on $[0,+\infty[$ is generated by the exponential distribution.

$$dF(x) = \xi e^{-\xi x} d\lambda(x)$$

The orthogonal polynomials linked to the exponential measure are the Laguerre ones, see Szegö (1939) [9]

- Which value for ξ to complete the integrability condition ?
- $\psi(u) \leq e^{-\gamma u}$.
  where γ is the unique positive solution to the following equation :

$$\widehat{F_{U^I}}(s) = \frac{1}{\rho}$$

$\hookrightarrow \ \xi < 2\gamma$

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Calibration des simulations

Regarding the ruin model, we assume that :

- The premium rate p is equal to 1,
- The safety loading η is equal to 20%.

A graphic visualisation is proposed, we plot the quantity :

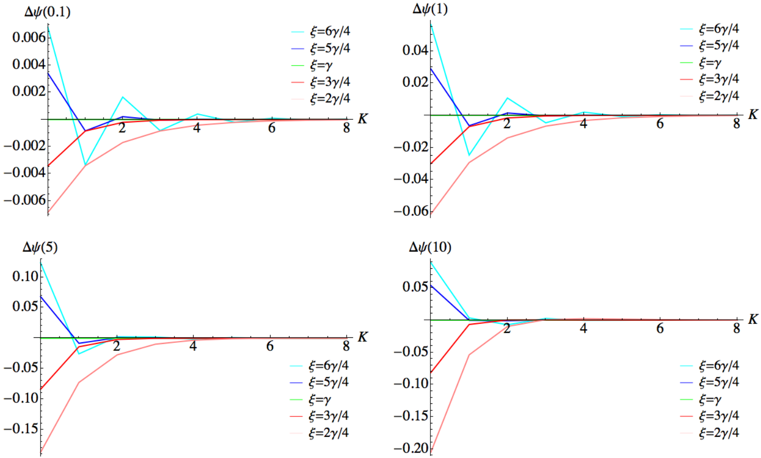
$$\Delta\psi(u) = \psi(u) - \psi^K(u),$$

for an intitial reserves u and a truncation order K.

$\hookrightarrow$ Different values for ξ are tested with one equal to γ.

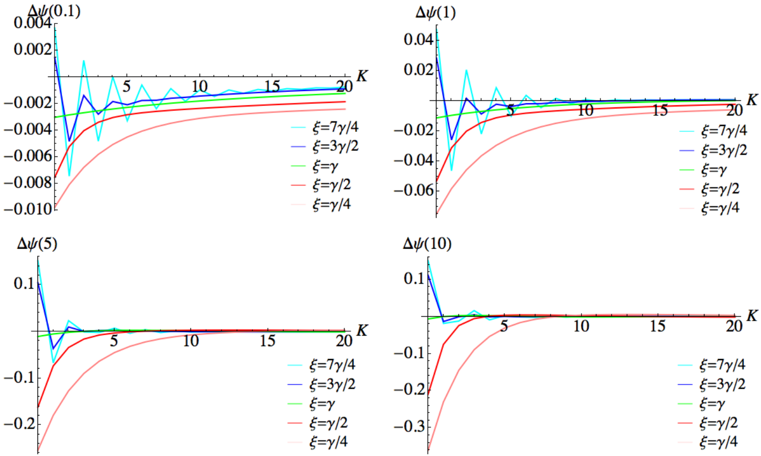
Exponentially distributed claim sizes

$$f_U(x) = e^{-x} 1_{\mathbb{R}^+}(x)$$



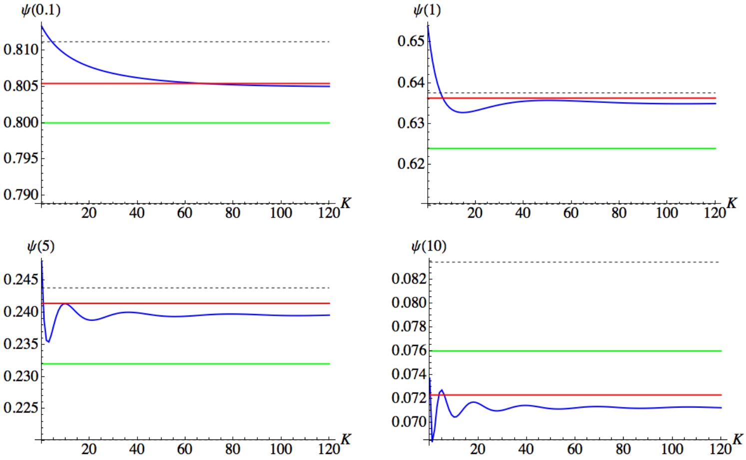
$\Gamma(1/2, 1/2) distributedclaimsizes$

$$f_U(x) = \frac{e^{-x/2}}{\Gamma(1/2)\sqrt{2x}} 1_{\mathbb{R}^+}(x)$$



$\Gamma(1/3,1) distributedclaimsizes$

$$f_U(x) = \frac{e^{-x}x^{-2/3}}{\Gamma(1/3)} 1_{\mathbb{R}^+}(x)$$



Comparison with Panjer's algorithm

u	Exact Value	Polynomials expansion $\xi = \gamma$, K=120	Panjer's algorithm h=0.01
0.1	0.821313	0.821424	0.821356
1	0.736114	0.736238	0.736395
5	0.47301	0.472944	0.473757
10	0.274299	0.274252	0.275131
50	0.00352109	0.00352476	0.00357292

u	Monte-Carlo simulations	Polynomials expansion $\xi = \gamma$, K=120	Panjer's algorithm h=0.01
0.1	0.8	0.80505	0.805454
1	0.624	0.634979	0.636315
5	0.232	0.239601	0.241442
10	0.076	0.0712518	0.0723159
50	0	4.569555×10^{-6}	4.686×10^{-6}

Conclusion

+ An efficient numerical method
  - $\hookrightarrow$ An approximation as precise as one might want
+ Easy to understand and to implement
+ No discretization of the claim sizes is needed

− Limited to light tailed distribution

Outlooks :

- Theoritivcal sensitiveness study of the parameter ξ
- Agregate claim amounts distribution, more general compound distributions
- Statistical extension
  - $\rightarrow$ Extension statistique
- Finite time ruin probability

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