Boundary crossing problems
Applications to Risk Management

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Insurance Risk Model

Consider a non-life insurance company.

- Up to some time horizon t :
  - $\hookrightarrow$ The number of claims is modeled through a counting process $\{N(t)\ ;\ t \geq 0\}$,
  - $\hookrightarrow$ The claim sizes are a sequence of non-negative, **i.i.d.** random variables $\{U_k\ ;\ k \in \mathbb{N}\}$,
  - $\hookrightarrow$ Initial capital of amount $u \geq 0$,
  - $\hookrightarrow$ Premium are collected linearly in time at a rate $c \geq 0$.
- The insurance risk reserve process is given by

$$R(t) = u + ct - \sum_{k=1}^{N(t)} U_k$$

Finite time ruin probability in the insurance ruin model

The time to ruin is defined as

$$\tau_u = \inf\{t \geq 0 \; ; \; R(t) < 0\}$$

and the finite time ruin probability

$$\psi(u, t) = \mathbb{P}(\tau_u \leq t)$$

- What's the point of computing such a quantity?

Theorem

Finite time non-ruin probability

If

- $u = 0$,
- $U_k = 1$ a.s.,
- $\{N(t)\ ;\ t \geq 0\}$ is a homogenous Poisson process
  - $\hookrightarrow$ The risk process is simply given by $R(t) = ct - N(t),\ \text{for } t \geq 0$

and we have

$$\mathbb{P}(\tau_0 > t) = \mathbb{E}\left[\left(1 - \frac{N(t)}{ct}\right)_+\right]$$

About the Poisson process

- A homogeneous poisson process is characterized by its inter arrival times $\{\Delta T_k \ ; \ k \geq 1\}$
  - $\hookrightarrow$ **i.i.d.** exponentially distributed with parameter λ
- Interesting property upon the jump times $\{T_n = \sum_{k=1}^{n} \Delta T_k \ ; \ n \geq 1\}$

$$[T_1, \ldots, T_n | N(t) = n] \stackrel{\mathcal{D}}{=} [\mathcal{U}_{1;n}(0,t), \ldots, \mathcal{U}_{n;n}(0,t)]$$

$\hookrightarrow$ *Noter au tableau.*

Appell sequence of polynomials

- $U = \{u_k \ ; \ k \geq 1\}$ is a sequence of real number.
- $\{A_k(x|U) \ ; \ k \geq 1\}$ is a sequence of polynomials such that

$$\begin{aligned} A_0(x|U) &= 1 \\ A_n'(x|U) &= n \times A_{n-1}(x|U) \text{ for } n > 1 \\ A_n(u_n|U) &= 0. \end{aligned}$$

Therefore

$$A_n(x|U) = n \int_{u_n}^{x} A_{n-1}(y_n|U) \mathrm{d}y_n.$$

and by induction we have the integral representation

$$A_n(x|U) = n! \int_{u_n}^{x} \int_{u_{n-1}}^{y_n} \ldots \int_{u_1}^{y_2} \mathrm{d}y_1 \ldots \mathrm{d}y_n.$$

Probabilistic interpretation of Appell polynomials

Because of

$$f_{U_{1:n},\ldots,U_{n:n}}(u_1,\ldots,u_n) = n!\mathbb{I}_{\{0<u_1\leq\ldots\leq u_n\leq 1\}} \tag{1}$$

where $U_{1:n},\ldots,U_{n:n}$ are the order statistics of a random sample of size n uniformly distributed.

- Up to some ordering conditions over U,

$$\begin{aligned} A_n(x|U) &= \mathbb{P}(U_{1:n}\geq u_1,\ldots,U_{n:n}\geq u_n, \text{ and } U_{n:n}\leq x) \\ A_n(1|U) &= \mathbb{P}(U_{1:n}\geq u_1,\ldots,U_{n:n}\geq u_n) \end{aligned}$$

$\hookrightarrow$ *Noter au tableau.*

Useful Identities

Let us have $bU = \{bu_k \; ; \; k \geq 1\}$, for $b \neq 0$, then

$$A_n(x|bU) = b^n A_n\left(\frac{x}{b}\middle|U\right) \; \#\text{Property1.}$$

And

$$A_n(x|1,\ldots,n) = x^{n-1}(x-n) \; \#\text{Property2}$$

These identities follow from induction based on the previous integral representation.

$\hookrightarrow$ *Noter au tableau.*

Conditionning on the values of N(t)

$$\tau_0 = \inf\{t \geq 0 \; ; \; ct - N(t) \leq 0\} = \inf\{t \geq 0 \; ; \; N(t) \geq ct\}$$

$$\{\tau_0 \geq t\} = \bigcup_{n=0}^{+\infty} \{\tau_0 \geq t\} \cap \{N(t) = n\}$$

that is equivalent to

$$\{\tau_0 \geq t\} = \bigcup_{n=0}^{\lfloor ct \rfloor} \bigcap_{k=1}^{n} \left\{ T_k \geq \frac{k}{c} \right\} \cap \{N(t) = n\}$$

Applying Bayes Theorem,

$$\mathbb{P}\left(\tau_0 > t\right) = \sum_{n=0}^{\lfloor ct \rfloor} \mathbb{P}\left(\bigcap_{k=1}^{n} \left\{ T_k \geq \frac{k}{c} \right\} \Big| N(t) = n \right) \mathbb{P}(N(t) = n).$$

Dealing with conditionnal probability, the link toward Appell Polynomials

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{k=1}^{n}\left\{T_k \geq \frac{k}{c}\right\}\Big| N(t)=n\right) &= \mathbb{P}\left(\bigcap_{k=1}^{n}\left\{U_{k:n}(0,t) \geq \frac{k}{c}\right\}\right) \\
&= \mathbb{P}\left(\bigcap_{k=1}^{n}\left\{U_{k:n} \geq \frac{k}{ct}\right\}\right) \\
&= A_n\left(1 \Big| \frac{1}{ct}, \dots, \frac{n}{ct}\right)
\end{aligned}$$

Taking advantage of the algebraic properties of Appell Polynomials

$$
\begin{aligned}
A_n\left(1 \middle| \frac{1}{ct}, \ldots, \frac{n}{ct}\right) &= \left(\frac{1}{ct}\right)^n A_n\left(ct \middle| 1, \ldots, n\right) \;\; \#\text{Property1} \\
&= \left(\frac{1}{ct}\right)^n (ct - n)(ct)^{n-1} \;\; \#\text{Property2} \\
&= \left(1 - \frac{n}{ct}\right)
\end{aligned}
$$

Reinjection leads to

$$
\begin{aligned}
\mathbb{P}\left(\tau_0 > t\right) &= \sum_{n=0}^{\lfloor ct \rfloor} \left(1 - \frac{n}{ct}\right) \mathbb{P}(N(t) = n) \\
&= \mathbb{E}\left\{\left[1 - \frac{N(t)}{ct}\right]_+\right\}
\end{aligned}
$$

To go Further

- Extension to Order Statistic Point Processes
  - $\hookrightarrow$ Mixed Poisson process with a time transformation
  - $\hookrightarrow$ Mixed Sample process
- Algrebraic properties when U is a sequence of partial sums $\{\sum_{k=1}^{n} U_k \ ; \ n \geq 1\}$ where the U_k's are **i.i.d.**

Dual Risk Model

Consider an investment company,

- Up to some time horizon s :
  - $\hookrightarrow$ The number of capital gains is modeled through a counting process $\{M(s)\ ;\ s \geq 0\}$,
  - $\hookrightarrow$ The capital gains are a sequence of non-negative, **i.i.d.** random variables $\{V_k\ ;\ k \in \mathbb{N}\}$,
  - $\hookrightarrow$ Initial capital of amount $v \geq 0$,
  - $\hookrightarrow$ Operational expenses entails a linearly decreasing financial reserve in time at a rate $d \geq 0$.
- The dual risk reserve process is given by

$$U(s) = v - ds + \sum_{k=1}^{M(s)} V_k$$