Boundary Crossing of Order Statistics Point Processes

Pierre-Olivier Goffard, Claude Lefèvre
Université Libre de Bruxelles, Département de Mathématique,
Campus de la Plaine C.P. 210, B-1050 Bruxelles, Belgium

Abstract

This paper is concerned with the first crossing of an order statistic point process through general moving boundaries. Our purpose is to determine exact boundary crossing probabilities in both one and two boundary cases. Simple recursive methods are obtained that exploit an underlying algebraic structure of polynomial type. This structure is a direct consequence of the order statistic property. The proposed approach is easy to implement and efficient. Perspectives in statistics are also announced.

MSC 2010: 60G55, 60G40, 12E10.
Keywords: Order statistic property; lower and upper boundaries; Appell and Abel-Gontcharoff polynomials.

1 Introduction

Many problems in stochastic modeling come down to study the crossing time of a certain stochastic process through a given boundary, lower or upper. Typical fields of application are in risk theory, epidemic modeling, queueing, reliability and sequential analysis. The present paper is concerned with the well-known class of point processes with the order statistic property (OSPP: Order Statistic Point Processes).

Definition 1.1. *A point process* $\{N(t),\, t \geq 0\}$ *with* $N(0) = 0$ *is an OSPP if for every* $n \geq 1$*, provided* $P(N(t) = n) > 0$*, then conditioned upon* $(N(t) = n)$*, the successive jump times* $(T_1, T_2, \ldots, T_n)$ *are distributed as the order statistics of* n *i.i.d. random variables with distribution function* $F_t(x)$ *supported on* $[0, t]$.

So, for any fixed time t, the only knowledge of the distribution of $N(t)$ and the distribution function $F_t(s)$, for $0 \leq s \leq t$, supplies a complete description of the process over $[0, t]$. Several authors have characterized the class of OSPP under certain conditions. The reader is referred to Crump [4], Feigin [5] and Puri [19]. We focus our interest on the first-crossing time of an OSPP through a one-sided boundary, lower $h_\alpha(t)$ or upper $h_\beta(t)$, and the first-exit time from a two-sided region delimited by two boundaries $h_\alpha(t)$ and $h_\beta(t)$ that are parallel. The boundaries are allowed to be moving, linearly or not. They are non-decreasing, which is not restrictive. We write them under the form

$$h_\alpha(t) = h(t) - \alpha,$$
$$h_\beta(t) = h(t) + \beta, \quad t \geq 0,$$

where α and β are two real numbers such that $\alpha > 0$ and $\beta \geq 0$ and $h(t)$ is a non-decreasing function with $h(0) = 0$.

In the one boundary case, we will obtain the exact distribution of the first meeting time τ_α in the lower boundary h_α:

$$\tau_\alpha = \inf\{t \geq 0 : N(t) = h_\alpha(t)\},$$

and the first crossing time τ_β through the upper boundary h_β:

$$\tau_\beta = \inf\{t \geq 0 : N(t) \geq h_\beta(t)\}.$$

For a region delimited by two such boundaries, we assume that h_α and h_β are parallel (i.e. share the same function h), and we derive the distribution of the first exit time from the region:

$$\tau_{\alpha,\beta} = \min(\tau_\alpha, \tau_\beta),$$

and the first exit time by the lower boundary h_α without crossing the upper boundary h_β:

$$\tau^*_{\alpha,\beta} = \inf\{t \geq 0 : N(t) = h_\alpha(t) \text{ and } N(s) < h_\beta(s) \,\text{for}\, s < t\}.$$

Crossing problems for Poisson and compound Poisson processes have received a considerable attention, especially for linear boundaries. We mention e.g. Zacks [23], Gallot [6], Picard and Lefèvre [12], Lehmann [15], Perry et al. [17], Stadje and Zacks [20], Ignatov and Kaishev [9], Lefèvre [10], Lefèvre and Loisel [11], Lefèvre and Picard [13], Xu [22]. These works were motivated by various topics including the ruin in insurance, the final size in epidemics, the busy period in queues or the detection of changes.

To obtain crossing probabilities for these processes, a method frequently used is to work with Laplace transforms. An alternative approach, proposed intially by Picard and Lefèvre [12], consists in applying simple recursive relations to calculate the probabilities. This is such a method that we choose to follow here. The recursions derived for the computations will rely on the existence of an underlying polynomial structure in the probabilities. The existence of this algebraic structure is a consequence of the order statistic property.

The formulas have been implemented in `Mathematica`. The source code is provided online at [7]. The recursive relations contribute to an efficient evaluation in terms of computing time. We invite the reader to appreciate the impact on the crossing probabilities when tuning the parameters of the OSPP or changing the shape of the boundary. A Monte Carlo estimator of the crossing probabilities is defined from our formulas and compared to a classical Monte Carlo estimator. The new Monte Carlo estimator beats the classical one in terms of variance. This paves the way for potential applications in a statistical framework.

The paper is organized as follows. Section 2 presents the class of OSPP under consideration and the families of polynomials used as a key tool in the study. Section 3 deals with the first meeting problem of an OSPP in a lower boundary. Section 4 examines the first crossing problem of an OSPP through an upper boundary. Section 5 considers the first exit problem for an OSPP that is trapped inside two boundaries. Section 6 discusses a Monte Carlo estimator constructed from the results derived before.

2 Framework

We start by presenting the general class of OSPP's with several standard particular cases and the different polynomial families that will be used in the analysis.

2.1 Order statistic point processes

A complete characterization of the OSPP's was obtained by Puri [19], following on earlier work. His main result is stated below (see [19, Theorem 5]).

Proposition 2.1. *Let $\{N(t),\, t \geq 0\}$ be an OSPP where $\mu(t) = E[N(t)]$ is finite for all t.*

1. *If $\lim_{t\to\infty} \mu(t) = \infty$, there exists a Poisson process $\{\mathcal{P}(t),\, t \geq 0\}$ with rate 1, an independent non-negative random variable W, both on the same space probability as is the OSPP, and a time transformation $\nu(t)$ such that*

$$N(t) = \mathcal{P}\left[W\nu(t)\right], \quad t \geq 0, \quad a.s. \tag{2.1}$$

   In other words, $\{N(t),\, t \geq 0\}$ is a mixed Poisson process up to a time-scale transformation.

2. *If $\lim_{t\to\infty} \mu(t) = \gamma < \infty$, there exists a process $\{\mathcal{D}_Z(t),\, t \geq 0\}$ that counts the deaths during $(0, t]$ in a process with initially Z individuals and i.i.d. lifetimes of distribution function $\mu(t)/\gamma$, $t \geq 0$, and an independent non-negative integer-valued random variable Z, both on the same probability space as is the OSPP, such that*

$$N(t) = \mathcal{D}_Z(t), \quad t \geq 0, \quad a.s. \tag{2.2}$$

   The process $\{\mathcal{D}_Z(t),\, t \geq 0\}$ is named a mixed sample process.

In both cases, the order statistic property holds over $(0, t]$ with

$$F_t(s) = \mu(s)/\mu(t), \quad 0 \leq s \leq t. \tag{2.3}$$

Note that $E[N(t)] = \mu(t)$ implies that $E(Z) = \gamma$ in (2.2). As an illustration, a few particular cases of OSPP of special interest are highlighted.

Special cases.

(i) A Poisson process of parameter $\lambda > 0$. $N(t)$ has a Poisson distribution of intensity λt and

$$F_t(s) = s/t, \quad 0 \leq s \leq t.$$

Formula (2.1) holds with $\nu(t) = t$ and $W = \lambda$.

(ii) A Pólya-Lundberg process of parameters $\lambda > 0$ and $b \geq 0$, i.e. a non-homogeneous linear birth process of rate $\lambda_n(t) = \lambda(1 + bn)/(1 + \lambda t)$. $N(t)$ has a negative binomial distribution:

$$P\left[N(t) = n\right] = \binom{n-1+1/b}{n} \left(\frac{\lambda bt}{1+\lambda bt}\right)^n \left(\frac{1}{1+\lambda bt}\right)^{1/b}, \quad n \geq 0.$$

and, here too, $F_t(s) = s/t$, for $0 \leq s \leq t$. Formula (2.1) holds with $\nu(t) = t$ and $W =_d \Gamma(1/b, \lambda b)$, where $\Gamma(r, m)$ denotes the gamma distribution with shape parameter $r > 0$ and mean parameter $m > 0$.

(iii) A linear birth process with immigration of birth rate $b > 0$ and immigration rate $\lambda \geq 0$. $N(t)$ has a negative binomial distribution:

$$P[N(t) = n] = e^{-\lambda t} \binom{\lambda/b + n - 1}{n} (1 - e^{-bt})^n, \quad n \geq 0,$$

and

$$F_t(s) = (e^{bs} - 1)/(e^{bt} - 1), \quad 0 \leq s \leq t.$$

Formula (2.1) holds with $\nu(t) = e^{bt} - 1$ and $W =_d \Gamma(\lambda/b, 1)$.

(iv) A death counting process in a linear death process of rate $b > 0$ and initial size $z \geq 1$. $N(t)$ has a binomial distribution:

$$P[N(t) = n] = \binom{z}{n} (1 - e^{-bt})^n e^{-bt(z-n)}, \quad 0 \leq n \leq z,$$

and

$$F_t(s) = (1 - e^{-bs})/(1 - e^{-bt}), \quad 0 \leq s \leq t.$$

Formula (2.2) holds with $\mu(t) = z(1 - e^{-bt})$, hence $\gamma = z$, and $Z = z$ almost surely.

2.2 Order statistics and polynomial structures

The first-crossing or exit problem of an OSPP will be formulated in terms of the order statistics $(U_{1:n}, \ldots, U_{n:n})$ for a sample of n uniform random variables on $(0, 1)$. It is known that the joint distributions of $(U_{1:n}, \ldots, U_{n:n})$ rely on an underlying polynomial structure (see Lefèvre and Picard [14] and the references therein). A few key points are recalled below.

2.2.1 One-sided joint distributions

Let $U = \{u_i,\, i \geq 1\}$ be a sequence of reals, non-decreasing in our context. To U is attached a (unique) family of Appell polynomials of degree n in x, $\{A_n(x|U),\, n \geq 0\}$ defined as follows.

Definition 2.2. *Starting with $A_0(x|U) = 1$, the $A_n(x|U)$'s satisfy the differential equations*

$$A_n^{(1)}(x|U) = nA_{n-1}(x|U), \tag{2.4}$$

with the border conditions

$$A_n(u_n|U) = 0, \quad n \geq 1. \tag{2.5}$$

So, each A_n has the integral representation

$$A_n(x|U) = n! \int_{u_n}^{x} \left[\int_{u_{n-1}}^{y_n} dy_{n-1} \ldots \int_{u_1}^{y_1} dy_1 \right] dy_n, \quad n \geq 1. \tag{2.6}$$

In parallel, to U is attached a (unique) family of Abel-Gontcharov (A-G) polynomials of degree n in x, $\{G_n(x|U),\, n \geq 0\}$.

Definition 2.3. *Starting with $G_0(x|U) = 1$, the $G_n(x|U)$'s satisfy the differential equations*

$$G_n^{(1)}(x|U) = nG_{n-1}(x|\mathcal{E}U), \tag{2.7}$$

where $\mathcal{E}U$ is the shifted family $\{u_{i+1},\, i \geq 1\}$, and with the border conditions

$$G_n(u_1|U) = 0, \quad n \geq 1. \tag{2.8}$$

So, each G_n has the integral representation

$$G_n(x|U) = n! \int_{u_1}^{x} \left[\int_{u_2}^{y_1} dy_2 \ldots \int_{u_n}^{y_{n-1}} dy_n \right] dy_1, \quad n \geq 1. \tag{2.9}$$

The Appell and A-G polynomials are closely related through the identity

$$G_n(x|u_1, \ldots, u_n) = A_n(x|u_n, \ldots, u_1), \quad n \geq 1. \tag{2.10}$$

The two families (i.e. for all $n \geq 0$), however, are distinct and enjoy different properties.

From (2.6) and (2.9), it is clear that the polynomials A_n and G_n can be interpreted in terms of the joint distribution of the vector $(U_{1:n}, \ldots, U_{n:n})$.

Proposition 2.4. *For $0 \leq u_1 \leq \ldots \leq u_n \leq x \leq 1$,*

$$P[U_{1:n} \geq u_1, \ldots, U_{n:n} \geq u_n \text{ and } U_{n:n} \leq x] = A_n(x|u_1, \ldots, u_n), \quad n \geq 1. \tag{2.11}$$

For $0 \leq x \leq u_1 \leq \ldots \leq u_n \leq 1$,

$$P[U_{1:n} \leq u_1, \ldots, U_{n:n} \leq u_n \text{ and } U_{1:n} \geq x] = (-1)^n\, G_n(x|u_1, \ldots, u_n), \quad n \geq 1. \tag{2.12}$$

The representations (2.11) and (2.12) will play a key role for the first-crossing of an OSPP through an upper and the first meeting of an OSPP and a lower boundary respectively. Numerically, it will be necessary to evaluate some special values of the polynomials. To this end, it is convenient to use the following recusive relations.

Proposition 2.5. *The Appell polynomials are computed through the expansion*

$$A_n(x|U) = \sum_{k=0}^{n} \binom{n}{k} A_{n-k}(0|U) x^k, \quad n \geq 1, \tag{2.13}$$

where the $A_n(0|U)$'s are obtained recursively from

$$A_n(0|U) = -\sum_{k=1}^{n} \binom{n}{k} A_{n-k}(0|U) u_n^k, \quad n \geq 1. \tag{2.14}$$

The A-G polynomials are computed through the recursion

$$G_n(x|U) = x^n - \sum_{k=0}^{n-1} \binom{n}{k} u_{k+1}^{n-k} G_k(x|U), \quad n \geq 1. \tag{2.15}$$

Formulas (2.13), and (2.14) follow from the Taylor's expansion of A_n, using also (2.4), and (2.5). Formula (2.15) follows from an Abelian expansion of x^n based on (2.7), and (2.8). Details are omitted here. Of course, the computing time increases with the degree of the polynomials.

Note that

$$A_n(x|a+bU) = b^n A_n\left((x-a)/b\,|U\right), \quad n \geq 1, \tag{2.16}$$

with the same identity for G_n. A simple special case is when the sequence U is affine.

Proposition 2.6. *If $U = \{u_i = a + b(i-1),\, i \geq 1\}$ for some reals a, b,*

$$A_n(x|U) = (x - a - b(n-1))(x - a + b)^{n-1}, \tag{2.17}$$

$$G_n(x|U) = (x-a)(x-a-bn)^{n-1}, \quad n \geq 1. \tag{2.18}$$

Formula (2.17) is proven by induction from (2.6). Formula (2.18) is then a consequence of (2.10). In addition to speeding up the computations, the identities (2.17) and (2.18) will allow us to link our results to some classical ones in the literature.

2.2.2 Two-sided joint distributions

Let $U = \{u_i,\, i \geq 1\}$ and $V = \{v_i,\, i \geq 1\}$ be two sequences of non-decreasing reals such that $0 \leq u_1 \leq \ldots \leq u_n \leq 1$, $0 \leq v_1 \leq \ldots \leq v_n \leq 1$ and $u_1 \leq v_1, \ldots, u_n \leq v_n$. Consider again the vector $(U_{1:n}, \ldots, U_{n:n})$. The joint distributions of interest are rectangular probabilities of the form

$$\begin{aligned} d_n(U,V) &= P[u_1 \leq U_{1:n} \leq v_1, \ldots, u_n \leq U_{n:n} \leq v_n] \\ &= n! \int_{u_n}^{v_n} \left[\int_{u_{n-1}}^{v_{n-1}\wedge y_n} \mathrm{d}y_{n-1} \ldots \int_{u_1}^{v_1 \wedge y_2} \mathrm{d}y_1 \right] \mathrm{d}y_n. \end{aligned} \tag{2.19}$$

A polynomial structure can be exhibited in this case too. The trick consists in varying the sample size, k say, of uniforms on $(0,1)$ from 1 to n. Specifically, the following rectangular probabilities are defined

$$d_k(U,V) = P[u_1 \leq U_{1:k} \leq v_1, \ldots, u_k \leq U_{k:k} \leq v_k], \quad 1 \leq k \leq n. \tag{2.20}$$

Proposition 2.7. *For $u_n \leq x \leq 1$,*

$$P[u_1 \leq U_{1:n} \leq v_1, \ldots, u_n \leq U_{n:n} \leq v_n \wedge x] = \sum_{k=0}^{n} \binom{n}{k} (v_{k+1} - x)_+^{n-k} (-1)^{n-k} d_k. \tag{2.21}$$

For instance, suppose that $u_n \leq v_{n-m}$ for some $m \geq 1$. Consider the intervals $[v_{n-i}, v_{n-i+1}]$ for $i = 0, \ldots, m$, with $v_{n+1} \equiv 1$. For $v_{n-i} \leq x \leq v_{n-i+1}$, (2.21) then reduces to

$$P[u_1 \leq U_{1:n} \leq v_1, \ldots, u_n \leq U_{n:n} \leq v_n \wedge x] = \sum_{k=n-i}^{n} \binom{n}{k} (x - v_{k+1})^{n-k} d_k,$$

which is a polynomial in x of degree i. So, for $u_n \leq x \leq 1$, (2.21) shows that the sought probabilty is a piecewise polynomial function. It has been named of Sheffer type by Niederhausen [16]. Now, let us take $x = u_n$ in (2.21). Evidently, the probability in the l.h.s. becomes 0. This provides us with the simple formula (2.22) below.

Proposition 2.8. *Let $d_0(U,V)=1$. The probabilities $d_n(U,V)$ are computed through the recursion*

$$d_n(U,V) = -\sum_{k=0}^{n-1} \binom{n}{k} (v_{k+1}-u_n)_+^{n-k} (-1)^{n-k} d_k(U,V), \quad n \geq 1. \tag{2.22}$$

3 First-meeting with a lower boundary

Let $\{N(t),\, t \geq 0\}$ be an OSPP that jumps at times T_n, $n \geq 1$ (see Definition 1.1). We recall that given $(N(t)=n)$, the vector $(T_1, \dots, T_n)$ is distributed as the order statistics of n i.i.d. random variables with distribution function $F_t(s)$, for $0 \leq s \leq t$. Consider for this process a lower boundary of the form $h_\alpha(t) = h(t) - \alpha$, where α is a non-negative real and h is a non-decreasing function with $h(0)=0$. We want to determine the distribution of the first-meeting time τ_α of the OSPP and this boundary. Let $h_\alpha^{-1}(x) = \inf\{t \geq 0 : h_\alpha(t) \geq x\}$ denote the generalized inverse of h_α. As the jumps of the process are of unitary height, the possible meeting levels are the naturals $0,1,2,\dots$. The corresponding meeting times are the instants $\alpha_n = h_\alpha^{-1}(n)$, $n \geq 0$, for which the boundary is integer-valued. This is illustrated in Figure 1 where the meeting happens at $\tau_\alpha = \alpha_5$.

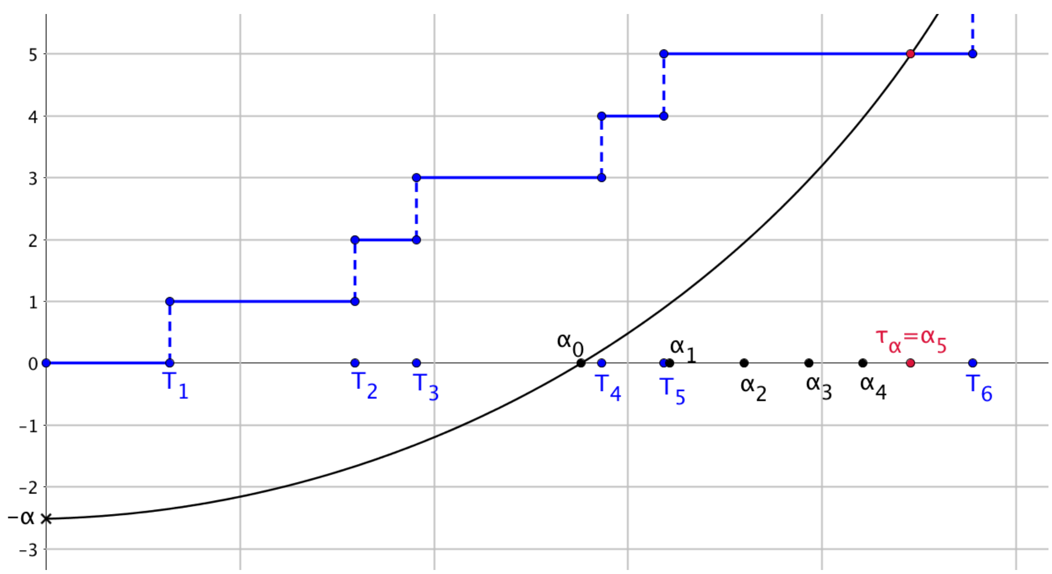

Figure 1: (blue, dashed) trajectory of the OSPP $\{N(t)\ ;\ t \geq 0\}$; (black, plain) lower boundary.

Proposition 3.1.

$$P(\tau_\alpha = \alpha_n) = (-1)^n\, P\left[N(\alpha_n)=n\right]\, G_n\left[0|\{F_{\alpha_n}(\alpha_0), \dots, F_{\alpha_n}(\alpha_{n-1})\}\right], \quad n \geq 0, \tag{3.1}$$

where $G_n\left[0|\{\dots\}\right]$ is an A-G polynomial such as defined in Subsection 2.2.1.

Proof. Obviously, $P(\tau_\alpha = \alpha_0) = P[N(\tau_\alpha)=0]$. For $n \geq 1$, the event $\{\tau_\alpha = \alpha_n\}$ can be expressed

$$\begin{aligned}\{\tau_\alpha = \alpha_n\} &= \{T_1 \leq \alpha_0\} \cap \{T_2 \leq \alpha_1\} \cap \dots \cap \{T_n \leq \alpha_{n-1}\} \cap \{T_{n+1} \geq \alpha_n\} \\ &= \bigcap_{i=1}^{n} \{T_i \leq \alpha_{i-1}\} \cap \{N(\alpha_n)=n\}.\end{aligned}$$

Conditioning on $\{N(\alpha_n)=n\}$, we then have

$$P(\tau_\alpha=\alpha_n)=P[N(\alpha_n)=n]\,P\left[\bigcap_{i=1}^{n}\{T_i\le\alpha_{i-1}\}|N(\alpha_n)=n\right]. \tag{3.2}$$

By the order statistic property, we know that

$$P\left[\bigcap_{i=1}^{n}\{T_i\le\alpha_{i-1}\}|N(\alpha_n)=n\right]=P\left[\bigcap_{i=1}^{n}\{V_{i:n}(\alpha_n)\le\alpha_{i-1}\}\right], \tag{3.3}$$

where $(V_{1:n}(\alpha_n),\ldots,V_{n:n}(\alpha_n))$ are the order statistics for n i.i.d. random variables of distribution function F_{α_n} defined on $(0,\alpha_n)$. Note that

$$F_{\alpha_n}(V_{i:n}(\alpha_n))=_d U_{i:n},\quad 1\le i\le n,$$

where $(U_{1:n},\ldots,U_{n:n})$ are the order statistics for n independent uniforms on $(0,1)$. Thus, (3.3) may be rewritten as

$$\begin{aligned}P\left[\bigcap_{i=1}^{n}\{T_i\le\alpha_{i-1}\}|N(\alpha_n)=n\right] &= P\left[\bigcap_{i=1}^{n}\{F_{\alpha_n}(V_{i:n}(\alpha_n))\le F_{\alpha_n}(\alpha_{i-1})\}\right]\\ &= P\left[\bigcap_{i=1}^{n}\{U_{i:n}\le F_{\alpha_n}(\alpha_{i-1})\}\right]\\ &= (-1)^nG_n\left[0|\{F_{\alpha_n}(\alpha_0),\ldots,F_{\alpha_n}(\alpha_{n-1})\}\right]. \end{aligned} \tag{3.4}$$

The last equality follows from formula (2.12) for the A-G polynomials. Combining (3.2), (3.3), (3.4) yields the announced result (3.1). □

Formula (3.1) has been implemented via `Mathematica`. The code is available online at [7]. The values of $G_n\left[0|\{\ldots\}\right]$ are computed using the recursion (2.15). The characterization of an OSPP (see Proposition 2.1) enables us to provide a more explicit version of (3.1).

Proposition 3.2. *1. If* $\lim_{t\to\infty}\mu(t)=\infty$,

$$P(\tau_\alpha=\alpha_n)=\frac{(-1)^n}{n!}\,G_n\left[0|\{\nu(\alpha_0),\ldots,\nu(\alpha_{n-1})\}\right]\,E[W^ne^{-W\nu(\alpha_n)}],\quad n\ge 0. \tag{3.5}$$

2. If $\lim_{t\to\infty}\mu(t)=\gamma<\infty$,

$$P(\tau_\alpha=\alpha_n)=\frac{(-1)^n}{\gamma^n}\,G_n\left[0|\mu(\alpha_0),\ldots,\mu(\alpha_{n-1})\right]\,E\left\{\binom{Z}{n}\left[1-\frac{\mu(\alpha_n)}{\gamma}\right]^{Z-n}\right\},\quad n\ge 0. \tag{3.6}$$

To obtain (3.5) and (3.6), it suffices to use in (3.1) the identity (2.16) for the A-G polynomials. For illustration, let us examine the special cases (i)-(iv) presented in section 2.1. Then, (3.1) and (2.16) give the following distributions for τ_α.

Corollary 3.3. *(i) For a Poisson process,*

$$P(\tau_\alpha = \alpha_n) = \frac{(-\lambda)^n}{n!} e^{-\lambda \alpha_n} G_n(0|\alpha_0, \ldots, \alpha_{n-1}), \quad n \geq 0,$$

(ii) a Pólya-Lundberg process,

$$P(\tau_\alpha = \alpha_n) = (-\lambda b)^n \binom{n-1+1/b}{n} \left(\frac{1}{1+\lambda b \alpha_n} \right)^{n+1/b} G_n(0|\alpha_0, \ldots, \alpha_{n-1}), \quad n \geq 0,$$

(iii) a linear birth process with immigration,

$$P(\tau_\alpha = \alpha_n) = (-1)^n \, e^{-(\lambda+bn)\alpha_n} \binom{\lambda/b+n-1}{n} G_n[1|\{e^{b\alpha_0}, \ldots, e^{b\alpha_{n-1}}\}], \quad n \geq 0,$$

(iv) a linear death counting process,

$$P(\tau_\alpha = \alpha_n) = e^{-(z-n)b\alpha_n} \binom{z}{n} G_n[1|\{e^{-b\alpha_0}, \ldots, e^{-b\alpha_{n-1}}\})], \quad 0 \leq n \leq z.$$

A simple situation is when $\{N(t),\, t \geq 0\}$ is a Poisson, or mixed Poisson, process and the lower boundary is linear.

Corollary 3.4. *For a (mixed) Poisson process, if $h_\alpha = ct - \alpha$ with $c > 0$,*

$$P(\tau_\alpha = \alpha_n) = \frac{\alpha_0}{\alpha_n} P[N(\alpha_n) = n], \quad n \geq 0. \tag{3.7}$$

Proof. If the OSPP is a mixed Poisson process, then $F_t(s) = s/t$, $0 \leq s \leq t$, so that (3.1) becomes

$$P(\tau_\alpha = \alpha_n) = (-1)^n \, P\left[N(\alpha_n) = n\right] (1/\alpha_n)^n \, G_n \left[0|\{\alpha_0, \ldots, \alpha_{n-1}\}\right], \quad n \geq 0. \tag{3.8}$$

Now, the function h_α being linear, its inverse is $h_\alpha^{-1}(x) = (\alpha + x)/c$. Thus, in (3.8), $\alpha_n = (\alpha + n)/c$, $n \geq 0$, i.e. α_n is an affine function of n. In such a case, G_n is given by formula (2.18), which yields

$$\begin{aligned} G_n \left[0|\{\alpha_0, \ldots, \alpha_{n-1}\}\right] &= G_n \left[0|\{(\alpha+i)/c,\, 0 \leq i \leq n-1\}\right] \\ &= (-1)^n (\alpha/c)[(\alpha+n)/c]^{n-1} \;=\; (-1)^n \alpha_0 \alpha_n^{n-1}. \end{aligned} \tag{3.9}$$

Inserting (3.9) in (3.8) then gives (3.7). $\diamond$

Formula (3.7) corresponds to a Kendall type identity (see e.g. Borovkov and Burq [3]). In fact, our main formula (3.1) can be viewed as a generalization of this identity to the first meeting problem of an OSPP through a moving lower boundary.

4 First-crossing through an upper boundary

Consider now the case of an upper boundary of the form $h_\beta(t) = h(t) + \beta$, where β is a positive real and h is a non-decreasing function with $h(0) = 0$. We denote by $\beta_n = h_\beta^{-1}(n)$, $n \geq 0$, the instant at which the upper barrier reaches the (integer-valued) level n, where h_β^{-1} is the generalized inverse of h_β. We want to determine the distribution of the first-crossing time τ_β of the OSPP through this boundary. Note that such a crossing will not occur through a meeting but by a jump over the boundary. This is illustrated in Figure 2 where the crossing occurs at $\tau_\beta = T_5$.

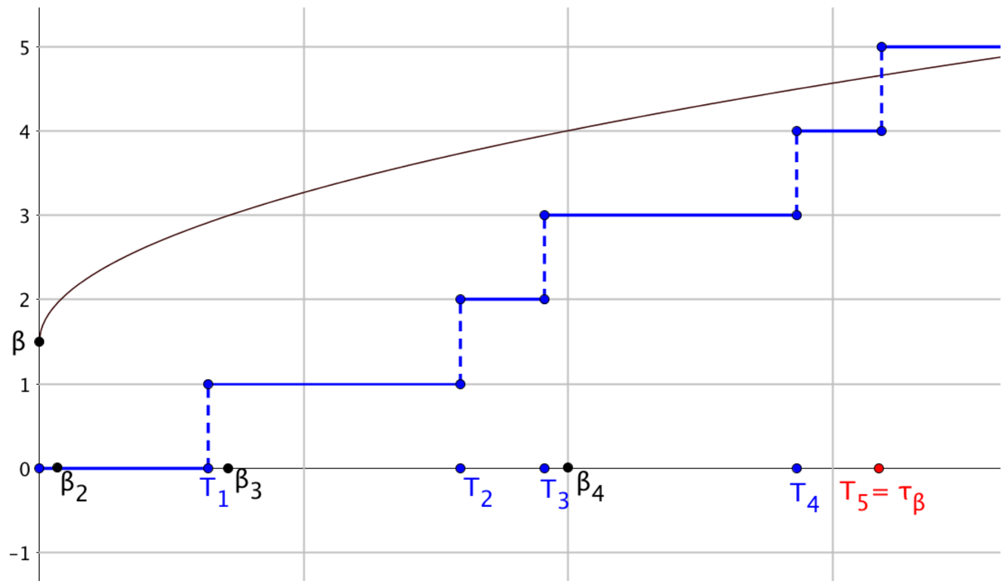

Figure 2: (blue, dashed) trajectory of the OSPP $\{N(t)\ ;\ t \geq 0\}$; (black, plain) upper boundary.

As the crossing is not a meeting, it is not convenient to work directly with the density function of τ_β. We will derive a formula for the survival function instead. Denote by $\mathbf{1}_A$ the indicator of an event A and by $\lfloor x \rfloor$ the integer part of a real x.

Proposition 4.1.

$$P(\tau_\beta > t) = E\left\{A_{N(t)}\left[1\middle|\{F_t(\beta_1), \ldots, F_t(\beta_{N(t)})\}\right] \mathbf{1}_{\{N(t) \leq \lfloor h_\beta(t) \rfloor\}}\right\}, \quad t \geq 0, \tag{4.1}$$

where $A_n(1|\{\ldots\})$ is an Appell polynomial such as defined in Subsection 2.2.1.

Proof. On the set $\{\tau_\beta > t\}$, the number of jumps of the process up to time t cannot exceed the level $\lfloor h_\beta(t) \rfloor$. Thus, we have

$$\begin{aligned}\{\tau_\beta > t\} &= \bigcup_{n=0}^{\lfloor h_\beta(t) \rfloor} \{N(t) = n\} \cap \{\tau_\beta > t\} \\ &= \bigcup_{n=0}^{\lfloor h_\beta(t) \rfloor} \{N(t) = n\} \cap \{\cap_{i=1}^n \{T_i > \beta_k\}\}.\end{aligned}$$

Conditionning on $\{N(t) = n\}$, we then get

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} P\left[\bigcap_{i=1}^n \{T_i > \beta_i\} | N(t) = n\right] P[N(t) = n], \tag{4.2}$$

and by the order statistic property,

$$P\left[\bigcap_{i=1}^n \{T_i > \beta_i\} | N(t) = n\right] = P\left[\bigcap_{i=1}^n \{V_{i:n}(t) > t_i\}\right], \tag{4.3}$$

where $(V_{1:n}(t), \ldots, V_{n:n}(t))$ are the order statistics for n i.i.d. random variables of distribution function F_t. Since

$$F_t[V_{i:n}(t)] =_d U_{i:n}, \quad 1 \leq i \leq n,$$

where $(U_{1:n}, \ldots, U_{n:n})$ are the order statistics for n independent uniforms on $(0,1)$, formula (4.3) may be rewritten as

$$\begin{aligned} P\left[\bigcap_{i=1}^{n}\{T_i > \beta_i\}|N(t) = n\right] &= P\left[\bigcap_{i=1}^{n}\{F_t[V_{i:n}(t)] > F_t(\beta_i)\}\right] \\ &= P\left[\bigcap_{i=1}^{n}\{U_{i:n} > F_t(\beta_i)\}\right] \\ &= A_n\left[1\middle|\{F_t(\beta_1), \ldots, F_t(\beta_n)\}\right]. \end{aligned} \tag{4.4}$$

Reinjecting (4.4) into (4.2) yields

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} A_n[1\big|\{F_t(\beta_1), \ldots, F_t(\beta_n)\}]\, P[N(t) = n],$$

which leads to the result (4.1). □

Here too, formula (4.1) has been implemented via `Mathematica`. The code is available online at [7]. Moreover, it is important to underline that the closed-form expression of (4.1) is quite suitable for an estimation by Monte Carlo simulation. In Section 6 we will show numerically that the Monte Carlo simulation of (4.1) leads to a less volatile estimator than the classical Monte Carlo estimator.

From Proposition 2.1 and using the identity (2.16), we can then deduce a more explicit version of formula (4.1).

Corollary 4.2. *1. If* $\lim_{t\to\infty} \mu(t) = \infty$,

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} A_n[\nu(t)|\{\nu(\beta_1), \ldots, \nu(\beta_n)\}]\, E\left[\frac{W^n e^{-W\nu(t)}}{n!}\right], \quad t \geq 0. \tag{4.5}$$

2. If $\lim_{t\to\infty} \mu(t) = \gamma < \infty$,

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} \frac{1}{\gamma^n} A_n[\mu(t)|\{\mu(\beta_1), \ldots, \mu(\beta_n)\}]\, E\left\{\binom{Z}{n}\left[1 - \frac{\mu(t)}{\gamma}\right]^{Z-n}\right\}, \quad t \geq 0. \tag{4.6}$$

Let us examine the special cases (i)-(iv) presented in Section 2.1.

Corollary 4.3. *(i) For a Poisson process,*

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} \frac{\lambda^n e^{-\lambda t}}{n!} A_n[t\big|\{\beta_1, \ldots, \beta_n\}], \quad t \geq 0,$$

(ii) a Pólya-Lundberg process,

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} \binom{1/b+n-1}{n} \left(\frac{\lambda b}{1+\lambda bt}\right)^n \left(\frac{1}{1+\lambda bt}\right)^{1/b} A_n[t \big| \{\beta_1, \ldots, \beta_n\}], \quad t \geq 0,$$

(iii) a linear birth process with immigration,

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} (-1)^n \binom{\lambda/b+n-1}{n} e^{-t(\lambda+bn)} A_n[e^{bt}-2 \big| \{e^{b\beta_1}, \ldots, e^{b\beta_n}\}], \quad t \geq 0,$$

(iv) a linear death counting process,

$$P(\tau_\beta > t) = \sum_{n=0}^{\lfloor h_\beta(t) \rfloor} (-1)^n e^{-(z-n)bt} \binom{z}{n} A_n[e^{-bt}-2 \big| \{e^{-b\beta_1}, \ldots, e^{-b\beta_n}\}], \quad t \geq 0.$$

Formula (4.1) becomes extremely simple in the particular case where $\{N(t)\ ;\ t \geq 0\}$ is a Poisson, or mixed Poisson, process and the upper boundary is a diagonal line. Let x_+ denote the positive part of a real x.

Corollary 4.4. *For a (mixed) Poisson process, if $h_\beta(t) \equiv h_0(t) = ct$ with $c > 0$,*

$$P(\tau_\beta > t) = E\left\{[1 - N(t)/ct]_+\right\}, \quad t \geq 0., \tag{4.7}$$

Proof. The OSPP being a mixed Poisson process, we have $F_t(s) = s/t$, $0 \leq s \leq t$, and since $h_\beta(t) \equiv h_0(t) = ct$, then $\beta_n \equiv 0_n = n/c$, $n \geq 0$. Thus, in (4.1),

$$\begin{aligned} A_{N(t)}\left[1 \big| \{F_t(\beta_1), \ldots, F_t(\beta_{N(t)})\}\right] &= A_{N(t)}\left[1 | \{i/ct,\, i \geq 1\}\right] \\ &= 1 - N(t)/ct, \end{aligned} \tag{4.8}$$

by virtue of formula (2.17). Inserting (4.8) in (4.1) then yields formula (4.7). □

A result of this form has a long history and different proofs are available. In particular, it has has been derived in the framework of ballot type theorems (e.g. Takács [21]). In risk theory, it corresponds to the finite time non-ruin probability in the compound Poisson risk model with no initial reserves and claim sizes equal to 1 (e.g. Asmussen and Albrecher [1, Theorem 2.1]).

5 First-exit with two parallel boundaries

Finally, we suppose that the OSPP is trapped into a region bounded from below by a lower boundary $h_\alpha(t) = h(t) - \alpha$ and from above by an upper boundary $h_\beta(t) = h(t) + \beta$, where $\alpha > 0$, $\beta \geq 0$ and the function h is non-decreasing with $h(0) = 0$. Observe that h_α and h_β are built with the same function h, i.e. the two boundaries are parallel. As before, we use the notation $\alpha_n = h_\alpha^{-1}(n)$, $n \geq 0$, and $\beta_n = h_\beta^{-1}(n)$, $n \geq 1$. Our aim is to derive the distribution of the first exit-time from this region, $\tau_{\alpha,\beta}$ say, and of the first-exit time through the lower boundary, $\tau^*_{\alpha,\beta}$ say. This is illustrated in Figure 3 where $\tau_{\alpha,\beta} = \tau^*_{\alpha,\beta} = \alpha_5$.

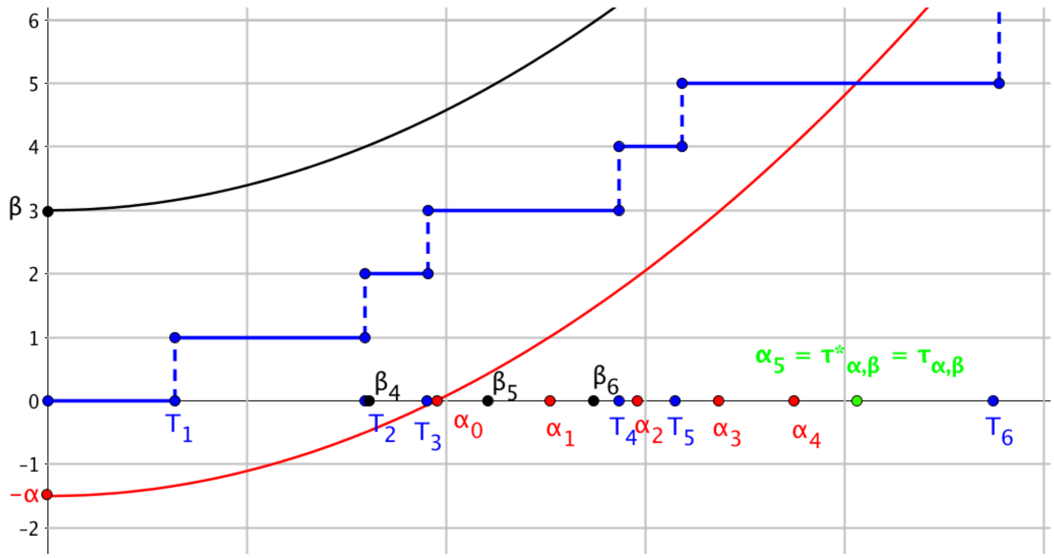

Figure 3: (blue, dashed) trajectory of the OSPP $\{N(t)\ ;\ t \geq 0\}$; (black, plain) upper boundary; (red, plain) lower boundary.

The assumption of parallel boundaries will allow us to deal with the class of OSPP processes. Let us mention that Xu [22] has examined the first-exit problem of a compound Poisson process with parallel linear boundaries. Applications of stopping times with parallel boundaries are also provided in that paper.

Proposition 5.1. *For $n \geq 1$,*

$$P\left(\tau^*_{\alpha,\beta} = \alpha_n\right) = d_n\left[\{F_{\alpha_n}(\beta_1), \ldots, F_{\alpha_n}(\beta_n)\}, \{F_{\alpha_n}(\alpha_0), \ldots, F_{\alpha_n}(\alpha_{n-1})\}\right] P[N(\alpha_n) = n], \tag{5.1}$$

and for $t \geq 0$,

$$P(\tau_{\alpha,\beta} > t) = E\left\{d_{N(t)}\left[\{F_t(\beta_1), \ldots, F_t(\beta_{N(t)})\}, \{F_t(\alpha_0), \ldots, F_t(\alpha_{N(t)-1})\}\right] \mathbf{1}_{\{\lfloor h_\alpha(t)\rfloor \leq N(t) \leq \lfloor h_\beta(t)\rfloor\}}\right\}, \tag{5.2}$$

where $d_n[\{\ldots\}, \{\ldots\}]$ is a rectangular probability such as defined in Subsection 2.2.2.

Proof. We proceed as for Propositions 3.1 and 4.1. First, consider the event $\{\tau^*_{\alpha,\beta} = \alpha_n\}$. For $n = 0$, it means $\{N(\alpha_0) = 0\}$. For $n \geq 1$, it is equivalent to

$$\begin{aligned}
\{\tau_\alpha = \alpha_n\} \cap \{\tau_\beta > \alpha_n\} &= \{\beta_1 \leq T_1 \leq \alpha_0\} \cap \ldots \cap \{\beta_n \leq T_n \leq \alpha_{n-1}\} \cap \{T_{n+1} \geq \alpha_n\} \\
&= \bigcap_{i=1}^{n} \{\beta_i \leq T_i \leq \alpha_{i-1}\} \cap \{N(\alpha_n) = n\}.
\end{aligned}$$

By conditioning on $\{N(\alpha_n) = n\}$ and using the order statistic property, we then have

$$\begin{aligned}
P\left(\tau^*_{\alpha,\beta} = \alpha_n\right) &= P\left[N(\alpha_n) = n\right] P\left[\bigcap_{i=1}^{n} \{\beta_i \leq T_i \leq \alpha_{i-1}\} | N(\alpha_n) = n\right] \\
&= P\left[N(\alpha_n) = n\right] P\left[\bigcap_{i=1}^{n} \{\beta_i \leq V_{i:n}(\alpha_n) \leq \alpha_{i-1}\}\right],
\end{aligned} \tag{5.3}$$

where $(V_{1:n}(\alpha_n), \dots, V_{n:n}(\alpha_n))$ are the order statistics for n i.i.d. random variables of distribution function F_{α_n}. Let us apply the transform F_{α_n} to get the order statistics $(U_{1:n}, \dots, U_{n:n})$ for n independent uniforms on $(0,1)$. Thus, we can write

$$\begin{aligned} P\left[\bigcap_{i=1}^{n}\{\beta_i \le V_{i:n}(\alpha_n) \le \alpha_{i-1}\}\right] &= P\left[\bigcap_{i=1}^{n}\{F_{\alpha_n}(\beta_i) \le U_{i:n} \le F_{\alpha_n}(\alpha_{i-1})\}\right] \\ &= d_n\left[\{F_{\alpha_n}(\beta_1), \dots, F_{\alpha_n}(\beta_n)\}, \{F_{\alpha_n}(\alpha_0), \dots, F_{\alpha_n}(\alpha_{n-1})\}\right], \end{aligned} \tag{5.4}$$

by virtue of definition (2.20). Combining (5.3) and (5.4) gives the result (5.1).

Now, consider the event $\{\tau_{\alpha,\beta} > t\} = \{\tau_\alpha > t\} \cap \{\tau_\beta > t\}$. We see that it can be expressed as

$$\begin{aligned} \{\tau_{\alpha,\beta} > t\} &= \bigcup_{n=\lfloor h_\alpha(t)\rfloor}^{\lfloor h_\beta(t)\rfloor} \{N(t) = n\} \cap \{\tau_\alpha > t\} \cap \{\tau_\beta > t\} \\ &= \bigcup_{n=\lfloor h_\alpha(t)\rfloor}^{\lfloor h_\beta(t)\rfloor} \{N(t) = n\} \cap \left\{\bigcap_{i=1}^{n}\{\beta_i \le T_i \le \alpha_{i-1}\}\right\}. \end{aligned}$$

Arguing as above, we then obtain

$$\begin{aligned} P(\tau_{\alpha,\beta} > t) &= \sum_{n=\lfloor h_\alpha(t)\rfloor}^{\lfloor h_\beta(t)\rfloor} P\left[\bigcap_{i=1}^{n}\{\beta_i \le T_i \le \alpha_{i-1}\} | N(t) = n\right] P[N(t) = n] \\ &= \sum_{n=\lfloor h_\alpha(t)\rfloor}^{\lfloor h_\beta(t)\rfloor} d_{N(t)}\left[\{F_t(\beta_1), \dots, F_t(\beta_{N(t)})\}, \{F_t(\alpha_0), \dots, F_t(\alpha_{N(t)-1})\}\right] P[N(t) = n], \end{aligned}$$

which is formula (5.2). □

First-exit problems with two boundaries have been less studied in the literature. In the case of the (compound) Poisson process, some interesting papers are e.g. Xu [22] cited before, Perry et al. [18] (in which the upper boundary is random) and Lehmann [15] (in which the boundaries are not parallel). The approach in these works is different from that followed here and relies on the specific structure of the Poisson process. For the general class of OSPP, the question of two arbitrary boundaries requires further investigation.

6 An induced simulation study

Our aim in this Section is to discuss an evaluation by simulation of the survival function of τ_β, the first-crossing time through an upper boundary. We recall that by Proposition 4.1,

$$\begin{aligned} P(\tau_\beta > t) &= E\left\{A_{N(t)}\left[1 \middle| \{F_t(\beta_1), \dots, F_t(\beta_{N(t)})\}\right] \mathbf{1}_{\{N(t) \le \lfloor h_\beta(t)\rfloor\}}\right\} \\ &\equiv E\left[p_{N(t)}\right] \ \text{say}, \quad t \ge 0. \end{aligned} \tag{6.1}$$

The survival function being expressed as an expectation, a Monte Carlo simulation is suitable for its estimation. We name it an Appell Polynomial Monte Carlo (APMC) procedure. We will show below the superiority of an APMC simulation over a Crude Monte Carlo (CMC) procedure. Note that a priori, an approximation by simulation could be viewed of limited relevance as formula (6.1) can be evaluated by recursion. Nevertheless, we will point out that our study paves the way for potential statistical applications.

For our problem, the CMC technique consists in writing the survival function of τ_β as

$$P(\tau_\beta > t) = E\left(\mathbf{1}_{\tau_\beta > t}\right), \quad t \geq 0. \tag{6.2}$$

The expectation in (6.2) is then approximated by the common Monte Carlo estimator based on trajectories drawn for the OSPP up to time t. For that, replications of $N(t)$ are simulated and the jump times are then given by the order statistics of a sample of size $N(t)$ random variables with distribution function F_t. The CMC evaluation of (6.2) falls in the rare event simulation framework where the quantity of interest is of the same magnitude as the variance of the CMC estimator, which makes it inefficient from a point of view of the committed relative error. For a detailed account on the CMC procedure and rare event simulations, see e.g. the book of Asmussen and Glynn [2].

Denote $p_t = P(\tau_\beta > t)$. The variance of the CMC estimator is given by

$$\sigma^2_{\mathrm{CMC}} = p_t(1 - p_t). \tag{6.3}$$

By comparison, the variance of the APMC estimator is

$$\begin{aligned}
\sigma^2_{APMC} &= Var\left\{A_{N(t)}\left[1\middle|\{F_t(\beta_1), \ldots, F_t(\beta_{N(t)})\}\right] \mathbf{1}_{\{N(t) \leq \lfloor h_\beta(t) \rfloor\}}\right\} \\
&= \sum_{n=0}^{\lfloor h(t)+\beta \rfloor} A_n[1|\{F_t(\beta_1), \ldots, F_t(\beta_n)\}]^2 P\left[N(t) = n\right] - p_t^2 \\
&\leq p_t(1 - p_t) = \sigma^2_{CMC}.
\end{aligned}$$

Thus, the APMC procedure enables us to reduce the variance of the estimator. The magnitude of the variance reduction is not easy to capture and depends on the situation.

For illustration, consider a Pólya-Lundberg process where $W =_d \Gamma(2, 1)$ (see (ii) in Subsection 2.1) and a polynomial upper boundary given by

$$h_\beta(t) = h(t) + \beta = t^2 + 3/2, \quad t \geq 0,$$

The probability of interest is $p_2 = P(\tau_\beta > 2) = 0.568265$. The variance of the APMC estimator is 0.184338 while the variance of the CMC estimator is 0.24534. So, the variance reduction when choosing the APMC procedure is arround 25%.

To explain a variance reduction, let us suppose that 5 trajectories are drawn for the process; they are displayed in Figure 4 (with the upper boundary). In general, the trajectories fall in three possible categories:

1. those ending below β for which both $p_{N(t)} = \mathbf{1}_{\tau_\beta > t} = 1$ since there is no crossing,
2. those ending above $h(t) + \beta$ for which both $p_{N(t)} = \mathbf{1}_{\tau_\beta > t} = 0$ since crossing is a.s.,
3. those ending between β and $h(t) + \beta$ for which $p_{N(t)} \leq 1$ and $\mathbf{1}_{\tau_\beta > t} = 0$ or 1.

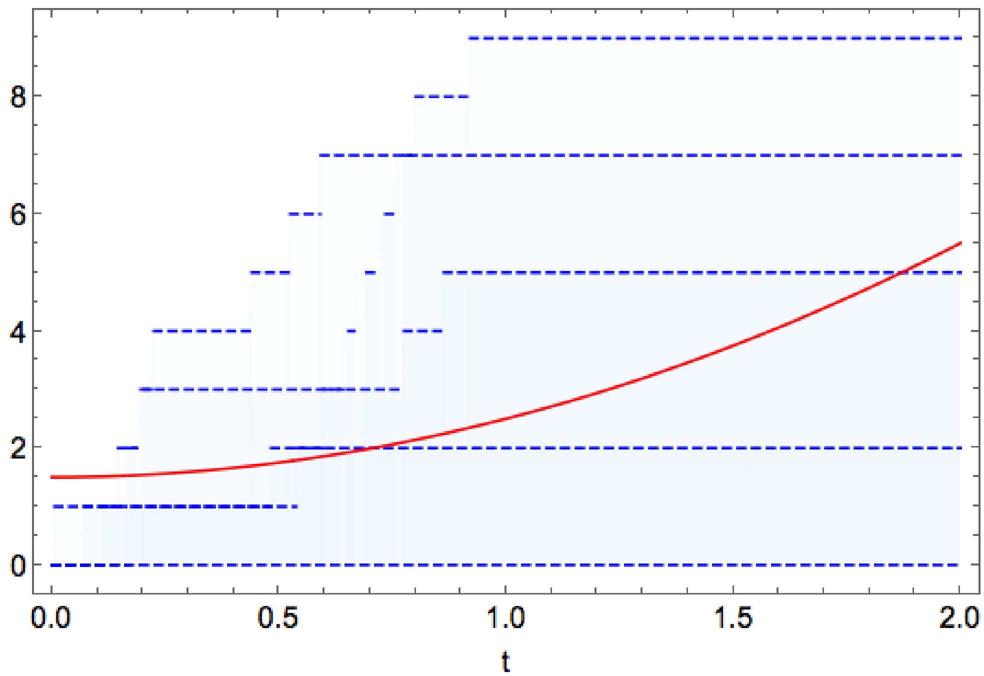

Figure 4: (blue, dashed) trajectories of the OSPP $\{N(t)\ ;\ t \geq 0\}$; (red, plain) upper boundary.

Trajectory	1	2	3	4	5
$p_{N(t)}$	0.875	0	1	0	0.0821616
$\mathbf{1}_{\tau_\beta > t}$	0	0	1	0	0

Table 1: Values of $\mathbf{1}_{\tau_\beta > t}$ and $p_{N(t)}$ for the 5 trajectories.

For the 5 trajectories in Figure 4, the values of $\mathbf{1}_{\tau_\beta > t}$ and $p_{N(t)}$ are given in Table 1. Intuitively, if many trajectories finish their run either below β or above $h(t) + \beta$, then $p_{N(t)} = \mathbf{1}_{\tau_\beta > t}$ very often and the variance reduction is not significant. On the contrary, when $N(t)$ is between β and $h(t) + \beta$, then the time at which the jumps occur plays an important role and the APMC simulation decreases the variance significantly. In practice, if $N(t) \in (\beta, h(t) + \beta)$, getting an approximation of $p_N(t)$ will imply repeated simulations of the jump times. This requires nested simulations that are known to be intense from a computational viewpoint. The variance reduction is expected to be larger when the likelihood for $N(t)$ of being between β and $h(t) + \beta$ is high. For the present example, the probability $P[N(t) \in (\beta, h(t) + \beta)]$ is increasing in t, as shown in Figure 5.

Moreover, in Figure 6, the variance of the CMC and APMC estimators is represented as function of t, as well as the relative difference, expressed as a percentage,

$$\Delta\sigma^2 = \frac{\sigma^2_{APMC} - \sigma^2_{CMC}}{\sigma^2_{CMC}}.$$

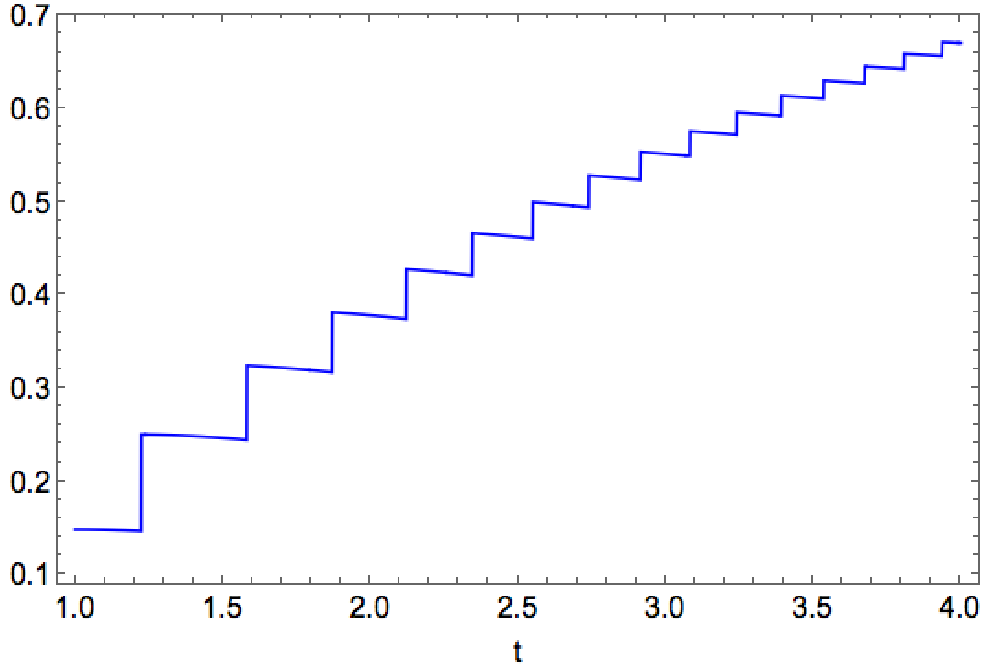

Figure 5: The probability $P[N(t) \in (\beta, h(t)+\beta)]$ as a function of t.

We observe that for $t < 1$, the variances are of similar size because the two estimators are rather close. For $t > 1$, the decay of the variance of the APMC estimator becomes clear. So, a variance reduction of more than 30% is reached when $t = 4$. Table 2 gives some numerical values for p_t, σ^2_{APMC}, σ^2_{CMC}, and $\Delta\sigma^2$ at different time horizons.

t	p_t	σ^2_{APMC}	σ^2_{CMC}	$\Delta\sigma^2$
1.	0.62963	0.196159	0.233196	-15.8824
2.	0.568265	0.184338	0.24534	-24.8643
3.	0.562963	0.167682	0.246036	-31.8464
4.	0.562619	0.158262	0.246079	-35.6867

Table 2: Values of p_t, σ^2_{APMC}, σ^2_{CMC}, and $\Delta\sigma^2$ at different time horizons.

However, a direct link between the probability $P[N(t) \in (\beta, h(t)+\beta)]$ and the variance reduction is not established so far. Changing the parameter values, the type of OSPP and the boundary shape allows us to appreciate the advantage of the APMC simulation. Further experiences can be conducted through the online accompaniment [7].

Let us add a remark on the use of formula (6.1) itself. This requires to determine the values of the Appell polynomials up to the degree $\lfloor h(t)+\beta \rfloor$, which is done by applying formula (2.13) with the recursion (2.14). These computations will be time consuming when $\lfloor h(t)+\beta \rfloor$ is large. However, if it is unlikely that the OSPP reaches the $\lfloor h(t)+\beta \rfloor$, the APMC estimator provides a way to reduce the computational effort as the Appell polynomials will be computed up to the degree

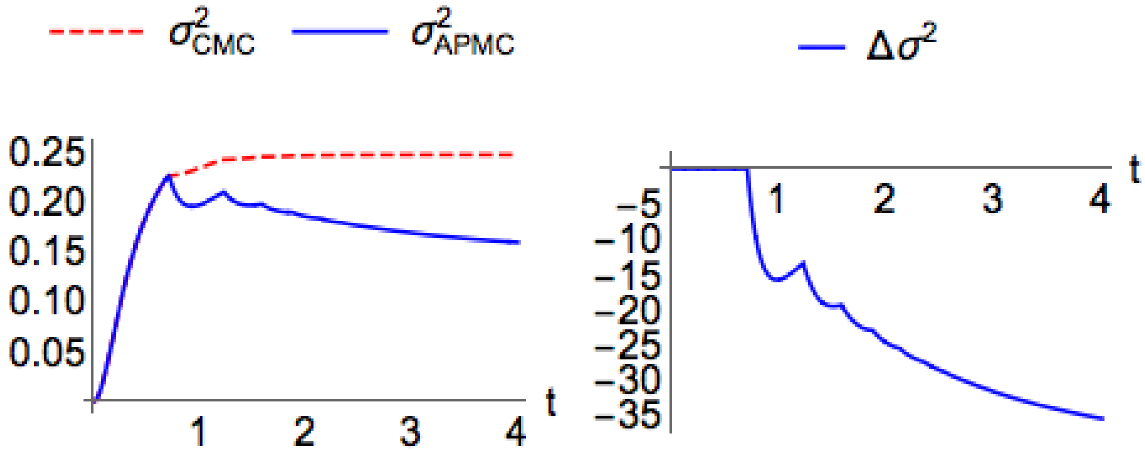

Figure 6: In funcrion of t, on the left side the variances σ^2_{CMC} (dashed) and σ^2_{APMC} (plain), on the right side the relative difference of variance $\Delta\sigma^2$.

given by $\min\{\lfloor h(t)+\beta\rfloor, N^*(t)\}$, where $N^*(t)$ is the maximum of the OSPP at time t over all the replications.

Statistical perspectives. To close, it is worth to emphasize potential applications in statistics. Suppose that a sample of trajectories up to a given time horizon is available. Each trajectory can be summarized by the jump times and the number of jumps. As the APMC estimator is less volatile than the CMC estimator, it leads to a more reliable estimation. What makes the APMC estimator even more appealing is that we do not need to keep track of the jump times to evaluate it, only the number of jumps is necessary. This is quite convenient because it is not always possible in practice to spot and report exactly when does an event occurs. In such a situation of incomplete information upon the observed trajectories, we can still propose an estimation of the probability under study. The cost is, of course, the assumption that the observed phenomenon is governed by an OSPP for which we have to identify the type and the parameters. One could argue that the mixed Poisson process emcompasses a broad class of stochastic processes and is well suited in many modelling problems (see the monograph by Grandell [8]).

Acknowledgement

We received support from the *ARC project IAPAS* of the Fédération Wallonie-Bruxelles.

References

[1] S. Asmussen and H. Albrecher. *Ruin Probabilities.* World Scientific, Singapore, 2010.

[2] S. Asmussen and P. W. Glynn. *Stochastic Simulation: Algorithms and Analysis*, volume 57 of *Stochastic Modelling and Applied Probability.* Springer, New York, 2007.

[3] K. Borovkov and Z. Burq. Kendall's identity for the first crossing time revisited. *Electronic Communication in Probability*, 6:91–94, 2001.

[4] K. S. Crump. On point processes having an order statistic structure. *Sankhyā: The Indian Journal of Statistics, Series A (1961-2002)*, 37(3):396–404, 1975.

[5] P. D. Feigin. On the characterization of point processes with the order statistic property. *Journal of Applied Probability*, 16(2):297–304, 1979.

[6] S. F. L. Gallot. Absorption and first-passage times for a compound poisson process in a general upper boundary. *Journal of Applied Probability*, 30(4):835–850, 1993.

[7] P.-O. Goffard and C. Lefèvre. *Online accompaniment for "Boundary Crossing of Order Statistics Point Process"*, 2016. https://github.com/LaGauffre/BoundaryCrossingOSPP.

[8] J. Grandell. *Mixed Poisson Process.* CRC Press, 1997.

[9] Z. G. Ignatov and V. K. Kaishev. A finite-time ruin probability formula for continuous claim severities. *Journal of Applied Probability*, 41(2):570–578, 2004.

[10] C. Lefèvre. Discrete compound Poisson process with curved boundaries: polynomial structures and recursions. *Methodology and Computing in Applied Probability*, 9(2):243–262, 2007.

[11] C. Lefèvre and S. Loisel. Finite-time ruin probabilities for discrete, possibly dependent, claim severities. *Methodology and Computing in Applied Probability*, 11(3):425–441, 2009.

[12] C. Lefèvre and P. Picard. First crossing of basic counting processes with lower non-linear boundaries: a unified approach through pseudopolynomials. *Advances in applied Probability*, 28(3):853–876, 1996.

[13] C. Lefèvre and P. Picard. A new look at the homogeneous risk model. *Insurance: Mathematics and Economics*, 49(3):512–519, 2011.

[14] C. Lefèvre and P. Picard. Risk models in insurance and epidemics: A bridge through randomized polynomials. *Probability in the Engineering and Informational Sciences*, 29:399–420, 2015.

[15] A. Lehmann. Boundary crossing probabilities of Poisson counting processes with general boundaries. *Advances in Stochastic Models for Reliability, Quality and Safety*, pages 153–166, 1998.

[16] H. Niederhausen. Sheffer polynomials for computing exact Kolmogorov-Smirnov and Rényi distributions. *The Annals of Statistics*, 9(5):923–944, 1981.

[17] D. Perry, W. Stadje, and S. Zacks. Contributions to the theory of firstexit times of some compound processes in queueing theory. *Queueing Systems*, 33(4):369–379, 1999.

[18] D. Perry, W. Stadje, and S. Zacks. A two-sided first-exit problem for a compound Poisson process with a random upper boundary. *Methodology and Computing in Applied Probability*, 7(1):51–62, 2005.

[19] P. S. Puri. On the characterization of point processes with the order statistic property without the moment condition. *Journal of Applied Probability*, 19(1):39–51, 1982.

[20] W. Stadje and S. Zacks. Upper first-exit times of compound Poisson processes revisited. *Probability in the Engineering and Informational Sciences*, 17:459–465, 2003.

[21] L. M. Takaćs. *Combinatorial Methods in the Theory of Stochastic Processes*, volume 126. Wiley, New York, 1967.

[22] Y. Xu. First exit times of compound Poisson processes with parallel boundaries. *Sequential Analysis*, 31(2):135–144, 2012.

[23] S. Zacks. Distributions of stopping times for Poisson processes with linear boundaries. *Stochastic Models*, 7(2):233–242, 1991.