Two numerical methods to evaluate stop-loss premiums

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October 30, 2017

Abstract

Two numerical methods are proposed to numerically evaluate the survival function of a compound distribution and the stop-loss premiums associated with a non-proportional global reinsurance treaty. The first method relies on a representation of the probability density function in terms of Laguerre polynomials and the gamma density, the second is a numerical inversion of the Laplace transform. Numerical comparisons are conducted at the end of the paper.

MSC 2010: 60G55, 60G40, 12E10.
Keywords: Risk theory; orthogonal polynomials; numerical Laplace transform inversion; reinsurance; stop-loss premium.

1 Introduction

Consider the random variable (**rv**)

$$S_N = \sum_{k=1}^{N} U_k,$$

where N is a counting **rv** and $\{U_k\}_{k\in\mathbb{N}_+}$ is a sequence of **rv's** which are independent and identically distributed (**iid**), non-negative, and independent of N. We denote the probability density function (**pdf**) of S_N as f_{S_N}, and its survival function (**sf**) as

$$\overline{F}_{S_N}(x) = \mathbb{P}(S_N > x), \quad \text{for } x \geq 0.$$

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This paper concerns approximations of f_{S_N} and $\overline{F}_{S_N}$, though we begin with a discussion of how S_N is used in actuarial science.

Frequently S_N models the aggregated losses of a non-life insurance portfolio over a given period of time—here N represents the number of claims and U_k the claim sizes—yet other applications also exist. Actuaries and risk managers typically want to quantify the risk of large losses by a single comprehensible number, a risk measure.

One popular risk measure is the Value-at-Risk (**VaR**). In actuarial contexts, the **VaR** at level $\alpha \in (0, 1)$ is defined such that the probability of (aggregated) losses exceeding the level **VaR** is at most $1 - \alpha$. We denote this α-quantile as

$$\mathrm{VaR}_{S_N}(\alpha) = \inf\{x \geq 0, F_{S_N}(x) \geq \alpha\}.$$

Following the European recommendation of the Solvency II directive, the standard value for α is 0.995, see [9]. It is used by risk managers in banks, insurance companies, and other financial institutions to allocate risk reserves and to determine solvency margins. Also, we have stop-loss premiums (**slp's**) which are risk measures that are commonly used in reinsurance agreements.

A reinsurance agreement is a common risk management contract between insurance companies, one called the "cedant" and the other the "reinsurer". Its aim is to keep the cedant's long-term earnings stable by protecting the cedant against large losses. The reinsurer absorbs part of the cedant's loss, say $f(S_N)$ where $0 \leq f(S_N) \leq S_N$, leaving the cedant with $I_f(S_N) = S_N - f(S_N)$. In return, the cedant pays a premium linked to

$$\Pi = \mathbb{E}[f(S_N)],$$

under the expected value premium principle.

In practice, there are a variety of reinsurance designs from which an insurer can choose. We focus in this work on the stop-loss reinsurance treaty associated with the following ceded loss function

$$f(S_N) = (S_N - a)_+, \ a \geq 0,$$

where a is referred to as the retention level or priority. The ratemaking of the stop-loss reinsurance policy requires the evaluation of

$$\Pi_a(S_N) = \mathbb{E}\left[(S_N - a)_+\right], \tag{1}$$

also known as the usual stop loss premium (**slp**).

One variation is the limited stop-loss function,

$$f(S_N) = \min[(S_N - a)_+, b], \ b \geq 0, \tag{2}$$

where b is called the limit. The limited stop-loss function (2) is very appealing in practice because it prevents the cedant from over-estimating their losses and therefore overcharging the reinsurer. We also have the change-loss function defined as

$$f(S_N) = c(S_N - a)_+, \ 0 \leq c \leq 1,$$

which is in between stop-loss and quota-share reinsurance. Note that the ratemaking in each case requires the expectation in (1).

From a practical point of view, a reinsurance treaty over the whole portfolio is less expensive to handle than one which involves claim-by-claim management. It also grants protection in the event of an unusual number of claims, triggered for instance by a natural disaster. From a theoretical point of view, it is well known that the stop-loss ceded function allows one to minimize the variance of the retained loss for a given premium level, see for instance the monograph of Denuit **et al.** [7]. Recently, it has been shown that stop-loss reinsurance is also optimal when trying to minimize the **VaR** and the expected shortfall of the retained loss, see the works of Cai **et al.** [4], Cheung [5], and Chi and Tan [6]. Note that some other ceded loss functions appear in their work, there are however very close to the stop-loss one.

Unfortunately, one is seriously constrained when calculating these quantities analytically, as there are only a few cases where either the **pdf** or the **sf** is available in a simple tractable form. To estimate the **VaR** or **slp** we must find fast and accurate approximations for these functions.

We discuss the use of an approximation of the **pdf** in terms of the gamma density and its orthonormal polynomials. This method has been studied in the recent works of Goffard **et al.** [13] and Jin **et al.** [17]. We emphasize here the computational aspect of this numerical method and detail some practical improvements. An exponential change of measure can be used to recover the **pdf** of S_N when the claim sizes are governed by a heavy-tailed distribution. This refinement has been successfully applied in the work of Asmussen **et al.** [3] to recover the density of the sum of lognormally distributed random variables.

This method is compared to a numerical inversion of the Laplace transform which is known to be efficient to recover the survival function of a compound distribution. The critical step in Laplace inversion is to select which numerical integration technique to apply. We implement a method inspired by the work of Abate and Whitt [1] which is very similar to the method of Rolski **et al.** [30, Chapter 5, Section 5]. An approximation of the **slp** is then proposed relying on the connection with the survival function of the equilibrium distribution of S_N. Note that Dufresne **et al.** [8] successfully applied a Laplace inversion based technique to the evaluation of **slp**.

To close, we want to emphasize the fact that the numerical methods also apply in a risk theory framework. The infinite-time ruin probability in the compound Poisson ruin model is equal to the survival function of a compound geometric distribution. The polynomial approximation and the Laplace inversion methods have been employed, and compared to solve this particular problem in the work of Goffard **et al.** [12]. We add a more original application by noting that the finite-time non-ruin probability with no initial reserves, again under the classical risk model assumptions, may be rewritten as the **slp** associated with a compound Poisson distribution where the priority is expressed in terms of the premium rate and the time horizon.

The rest of the paper is organized as follows. Section 2 introduces compound distributions, and details their role in risk theory. Section 3 presents the approximation method based on orthogonal polynomials. Section 4 presents the approximation through the numerical inversion of the Laplace transform. Section 5 is devoted to numerical illustrations where the performances of the two methods are compared; the MATHEMATICA code used in this section is available online [11].

2 Compound distributions and risk theory

After setting up some notational conventions for Laplace transforms, compound distributions are introduced along with a brief account of their importance in risk modeling.

2.1 Laplace transforms

Definition 1. *For a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, we define*

$$\mathcal{L}\{f\}(t) \equiv \int_0^\infty \mathrm{e}^{-tx} f(x)\,\mathrm{d}x\,, \quad \text{for } t \in \mathbb{C} \text{ with } \Re(t) \geq 0\,,$$

to be the corresponding Laplace transform. *For a positive random variable X with* ***pdf*** *f_X, we write $\mathcal{L}_X(t) \equiv \mathcal{L}\{f_X\}(t) = \mathbb{E}\,\mathrm{e}^{-tX}$.* ◇

We have the result that for $t > 0$

$$\mathcal{L}\{F_X\}(t) = \frac{\mathcal{L}\{f_X\}(t)}{t} = \frac{\mathcal{L}_X(t)}{t}\,, \text{ and}$$

$$\mathcal{L}\{\overline{F}_X\}(t) = \frac{1}{t} - \mathcal{L}\{F_X(x)\}(t) = \frac{1 - \mathcal{L}_X(t)}{t}\,.$$

2.2 Compound distribution

Let $S_N = \sum_{k=1}^N U_k$ be the aggregated claim amounts associated with a non-life insurance portfolio over a fixed time period. The number of claims, also called the claim frequency, is modeled by a counting random variable N having a probability mass function f_N. The claim sizes form a sequence $\{U_k\}_{k\in\mathbb{N}_+}$ of **iid** non-negative random variables with common **pdf** f_U. We further assume that the claim sizes are independent from the claim frequency, so the random variable S_N follows the compound distribution (f_N, f_U).

As $S_N = 0$ whenever $N = 0$ (assuming this occurs with positive probability), the distribution of S_N is the sum of a singular part (the probability mass $\mathbb{P}(S_N = 0) = f_N(0) > 0$) and a continuous part (describing S_N where $N > 0$) with a defective **pdf** $f^+_{S_N}$ and **cdf** $F^+_{S_N}$. From the law of total probability, we have

$$f^+_{S_N}(x) = \sum_{n=1}^{\infty} f_N(n) f_U^{*n}(x),\ x \geq 0. \tag{3}$$

This density is intractable because of the infinite series. Furthermore, the summands are defined by repeated convolution of f_U with itself which are rarely straightforward to evaluate. The methods presented in this work rely on the knowledge of the Laplace transform of S_N, given by

$$\mathcal{L}_{S_N}(t) = \mathcal{G}_N[\mathcal{L}_U(t)]\,,$$

where $\mathcal{G}_N(t) \equiv \mathbb{E}(t^N)$ is the *probability generating function* of N. The simple expression of the Laplace transform has made possible the use of numerical methods involving the moments or transform inversion to recover the distribution of S_N. The distribution is typically recovered using Panjer's algorithm or a Fast Fourier Transform algorithm based on the inversion of the discrete Fourier transform; these two methods are compared in the work of Embrechts and Frei [10]. Our orthogonal polynomial method involves the standard integer moment sequence for S_N, in contrast to more exotic types of moments used by some recent methods. Gzyl and Tagliani [15] uses the fractional moments within a max-entropic based method, while Mnatsakanov and Sarkisian [21] performs an inversion of the scaled Laplace transform via the exponential moments. In addition to proposing an approximation for the survival function of S_N, we provide an efficient way to compute the usual **slp** (1) for reinsurance applications.

2.3 Risk theory

In the classical risk model, the financial reserves of a non-life insurance company are modeled by the risk reserve process $\{R(t), t \geq 0\}$, defined as

$$R(t) = u + ct - \sum_{k=1}^{N(t)} U_k.$$

The insurance company holds an initial capital of amount $R(0) = u \geq 0$, and collects premiums at a constant rate of $c > 0$ per unit of time. The number of claims up to time $t \geq 0$ is governed by a homogeneous Poisson process $\{N(t), t \geq 0\}$ with intensity λ. The claim sizes are **iid** non-negative random variables independent from $N(t)$.

One of the goals of risk theory to evaluate an insurer's ruin probability, that is, the probability that the financial reserves eventually fall below zero. Of interest are both the finite-time ruin probability $\psi(u, T)$ and the infinite-time ruin probability, also called the *probability of ultimate ruin*, $\psi(u)$, which are defined as

$$\psi(u, T) = \mathbb{P}\Big(\inf_{0 \leq t \leq T} R(t) \leq 0\Big),$$

and

$$\psi(u) = \mathbb{P}\Big(\inf_{t \geq 0} R(t) \leq 0\Big).$$

These probabilities are often reformulated (for mathematical convenience) in terms of the associated claims surplus process $\{S(t), t \geq 0\}$,

$$S(t) = \sum_{k=1}^{N(t)} U_k - ct, \; t \geq 0,$$

specifically,

$$\psi(u, T) = \mathbb{P}\Big(\sup_{0 \leq t \leq T} S(t) \geq u\Big) \quad \text{and} \quad \psi(u) = \mathbb{P}\Big(\sup_{t \geq 0} S(t) \geq u\Big).$$

For a general background on risk theory and the evaluation of ruin probabilities, we refer the reader to the monograph of Asmussen and Albrecher [2].

The first connection between compound distributions and ruin probabilities is the following. If the net benefit condition is satisfied, **i.e.** if the premium rate exceeds the average cost of aggregated claims per unit of time, then the infinite-time ruin probability is given by the survival function of a geometric compound distribution. More precisely,

$$\psi(u) = \mathbb{P}\left(S_N \equiv \sum_{k=1}^{N} U_k^* > u\right) = (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{F}_{U^*}^{*n}(u),$$

with $N \sim \mathsf{Geom}_0(\rho)$, $\rho = \lambda\mathbb{E}(U)/c < 1$, and with **iid** U_k^* with **pdf** $f_{U^*}(x) = \overline{F}_U(x)/\mathbb{E}(U)$. This result is known as the Pollaczeck–Khinchine formula, see for instance Asmussen and Albrecher [2, Chapter IV, (2.2)]. Thus it is possible to evaluate the infinite-time ruin probability via Panjer's algorithm. If we are able to determine the Laplace transform of V then we can also apply the polynomial method of Goffard **et al.** [13], the fractional moment based method of Gzyl **et al.** [14], and the exponential moments based method of Mnatsakanov **et al.** [22].

The second connection links the finite-time ruin probability with no initial reserves to the **slp** associated with a compound distribution. If $N(t) \sim \mathsf{Poisson}(\lambda t)$ (**i.e.** claims arrive as a homogeneous Poisson process) then the finite-time ruin probability is given by

$$\psi(0,T) = 1 - \frac{1}{cT}\int_0^{cT} \mathbb{P}\Big(\sum_{i=1}^{N(T)} U_i \leq x\Big)\,\mathrm{d}x\,. \tag{4}$$

Writing $S_{N(T)} = \sum_{i=1}^{N(T)} U_i$ we can see that (4) says that $\psi(0,T) = \mathbb{E}[\min\{S_{N(T)}, cT\}]/cT$ and hence

$$\psi(0,T) = (cT)^{-1}\Big[\mathbb{E}[N(T)]\,\mathbb{E}[U_1] - \Pi_{cT}(S_{N(T)})\Big]\,. \tag{5}$$

Lefèvre and Picard [19, Corollary 4.3] show that equations (4) and (5) hold in the more general case where the claim arrival process forms a *mixed Poisson process*. This connection has been exploited recently in Lefèvre **et al.** [20] where the influence of the claim size distribution on the ruin probabilities is studied via stochastic ordering considerations.

3 Orthogonal polynomial approximations

3.1 Approximating general density functions

Let X be an arbitrary random variable with **pdf** [1] f_X with respect to some measure λ (typically Lebesgue measure on an interval or counting measure on a subset of $\mathbb{Z}$). We assume that the density is unknown and we propose an approximation of the form

$$\widehat{f}_X(x) = \sum_{k=0}^{K} q_k Q_k(x) f_\nu(x). \tag{6}$$

where f_ν is the reference or basis density, associated to a probability measure ν absolutely continuous with respect λ. The sequence $\{Q_k, k \geq 0\}$ is made of polynomials, orthonormal with respect to ν in the sense that

$$\langle Q_k, Q_l\rangle_\nu = \int Q_k(x)Q_l(x)\,\mathrm{d}\nu(x) = \delta_{k,l},\;\; k,l \in \mathbb{N}_0.$$

[1] This section is written from the perspective of approximating a **pdf**, however the main results also hold if applied to a defective density.

This sequence is generated by the Gram–Schmidt orthogonalization procedure which is only possible if ν admits moments of all orders. If additionally there exists $s > 0$ such that

$$\int e^{sx}\,d\nu(x) < \infty$$

then the sequence of polynomials $\{Q_k, k \geq 0\}$ forms an orthonormal basis of $L^2(\nu)$ which is the space of all square integrable functions with respect to ν, see the monograph by Nagy [25, Chapter 7]. Therefore, if $f_X/f_\nu \in L^2(\nu)$ then the polynomial representation of the density of X with respect to ν follows from orthogonal projection, namely we have

$$f_X(x)/f_\nu(x) = \sum_{k=0}^{\infty} \langle f_X/f_\nu, Q_k \rangle_\nu \, Q_k(x). \tag{7}$$

We label the coefficients of the expansion as $\{q_k, k \geq 0\}$, noting that

$$q_k \equiv \langle f_X/f_\nu, Q_k \rangle_\nu = \int Q_k(x) f_X(x) \frac{d\nu(x)}{f_\nu(x)} = \mathbb{E}\left[Q_k(X)\right], \ \ k \in \mathbb{N}_0,$$

and thus we can rearrange (7) to be

$$f_X(x) = \sum_{k=0}^{\infty} q_k Q_k(x) f_\nu(x). \tag{8}$$

The approximation (6) follows by simply truncating the series to $K+1$ terms.

The Parseval relationship, $\sum_{k=1}^{\infty} q_k^2 = ||f_X/f_\nu||_\nu^2$, ensures that the sequence $\{q_k, k \geq 0\}$ tends toward 0 as k tends to infinity. The accuracy of the approximation (6), for a given order of truncation K, depends on how swiftly these coefficients decay. The L^2 loss associated with the approximation of f_X/f_ν is $\sum_{k=K+1}^{\infty} q_k^2$.

Typical choices of reference distributions are ones that belong to a Natural Exponential Family with Quadratic Variance Function (NEF-QVF) which includes the normal, gamma, hyperbolic, Poisson, binomial, and Pascal distributions. This family of distributions is convenient as the associated orthogonal polynomials are classical, see the characterization by Morris [23]. The polynomials are known explicitly, thus we avoid the time-consuming Gram–Schmidt orthogonalization procedure. Furthermore, it has been shown in a paper by Provost [29] that the recovery of unknown densities from the knowledge of the moments of the distribution naturally leads to approximation in terms of the gamma density and Laguerre polynomials when X admits $\mathbb{R}_+$ as support, and in terms of the normal density and Hermite polynomials when X has $\mathbb{R}$ as support.

3.2 Approximating densities of positive random variables

To approximate the **pdf** for positive X, a natural candidate for the reference density is the gamma density. It has been proven to be efficient in practice, see the work of Goffard **et al.** [13, 12], and Jin **et al.** [17]. The work of Papush **et al.** [28] showed that among the gamma, normal and lognormal distribution, the gamma distribution seems to be better suited to model certain aggregate losses. The lognormal distribution is a problematic choice. Even though the orthogonal polynomials are available in a closed form see Asmussen **et al.**

[3], they do not provide a complete orthogonal system of the L^2 space. The case of the inverse Gaussian as basis received a treatment in the work of Nishii [26], where it is shown that the only way to get a complete system of polynomials is by using the Gram–Schmidt orthogonalization procedure. Differentiating the density (as it is done in the case of NEF-QVF) does not lead to an orthogonal polynomial system, and starting from the Laguerre polynomials leads to a system of orthogonal functions which is not complete. A solution might be to exploit the bi-orthogonality property pointed out in the work of Hassairi and Zarai [16]. To close this review of reference densities, we mention the work of Nadarajah **et al.** [24] where Weibull and exponentiated exponential distributions are considered as reference density.

The $\mathsf{Gamma}(r, m)$ distribution, where r is the shape parameter and m is the scale parameter, has a **pdf**

$$f_\nu(x) \equiv \gamma(r, m, x) = \frac{x^{r-1}\mathrm{e}^{-\frac{x}{m}}}{\Gamma(r)m^r},\ x \in \mathbb{R}^+,$$

where $\Gamma(\cdot)$ denotes the gamma function[2]. The associated orthonormal polynomials are given by

$$Q_n(x) = (-1)^n \binom{n+r-1}{n}^{-\frac{1}{2}} L_n^{r-1}\Big(\frac{x}{m}\Big) = (-1)^n \left(\frac{\Gamma(n+r)}{\Gamma(n+1)\Gamma(r)}\right)^{-\frac{1}{2}} L_n^{r-1}\Big(\frac{x}{m}\Big),$$

where $\{L_n^{r-1}, n \geq 0\}$ are the generalized Laguerre polynomials,

$$L_n^{r-1}(x) = \sum_{i=0}^{n} \binom{n+r-1}{n-i} \frac{(-x)^i}{i!} = \sum_{i=0}^{n} \frac{\Gamma(n+r)}{\Gamma(n-i+1)\Gamma(r+i)} \frac{(-x)^i}{i!},\ n \geq 0,$$

cf. the classical book by Szegö [31].

Lemma 1. *If ν is $\mathsf{Gamma}(r, m)$, the polynomial expansion (8) may be rewritten as*

$$f_X(x) = \sum_{i=0}^{\infty} p_i \gamma(r+i, m, x), \tag{9}$$

where

$$p_i = \sum_{k=i}^{\infty} q_k \frac{(-1)^{i+k}}{i!\,(k-i)!} \sqrt{\frac{k!\Gamma(k+r)}{\Gamma(r)}}, \tag{10}$$

and the function $\gamma(r, m, x)$ is the ***pdf*** *of the $\mathsf{Gamma}(r, m)$ distribution.*

Proof. If we change the sum in (8) from iterating over Laguerre polynomials to iterating over monomials we get

$$f_X(x) = \sum_{k=0}^{\infty} q_k Q_k(x) \gamma(r, m, x) = \sum_{i=0}^{\infty} c_i x^i \gamma(r, m, x)\,,$$

[2]For the distributions in this paper, we use Mathematica's parametrization, **e.g.** the exponential and Erlang distributions are $\mathsf{Exp}(\lambda) = \mathsf{Gamma}(1, 1/\lambda)$ and $\mathsf{Erlang}(n, m) = \mathsf{Gamma}(n, 1/m)$.

where

$$c_i = \sum_{k=0}^{\infty} \text{Coefficient}(x^i, q_k Q_k(x)) = \frac{(-1)^i}{m^i i!} \sum_{k=i}^{\infty} q_k (-1)^k \binom{k+r-1}{k}^{-\frac{1}{2}} \binom{k+r-1}{k-i}.$$

We also note that

$$x^i \gamma(r, m, x) = m^i \frac{\Gamma(r+i)}{\Gamma(r)} \gamma(r+i, m, x),$$

so

$$f_X(x) = \sum_{i=0}^{\infty} c_i m^i \frac{\Gamma(r+i)}{\Gamma(r)} \gamma(r+i, m, x) = \sum_{i=0}^{\infty} p_i \gamma(r+i, m, x),$$

where we have set $p_i = c_i m^i \Gamma(r+i)/\Gamma(r)$. □

Remark 3.1. *For $r = 1$, the formula for p_i, (10), simplifies to*

$$p_i = \sum_{k=i}^{\infty} q_k (-1)^{i+k} \binom{k}{i}.$$

The expression of the **pdf** in (9) resembles the one of an Erlang mixture, which are extensively used for risk modeling purposes, **cf.** Willmot and Woo [33], Lee and Lin [18], and Willmot and Lin [32]. However, the p_i's defined in (10) do not form a proper probability mass function as they are not always positive. Hence our approximation cannot be considered as an approximation through an Erlang mixture although it enjoys the same features when it comes to approximating the survival function and the **slp** as shown in the following result.

Proposition 1. *Letting $\Gamma_u(r, m, x)$ be the **sf** of the* Gamma(r, m) *distribution, we have:*

*(i) the **sf** of X is given by*

$$\overline{F}_X(x) = \sum_{i=0}^{\infty} p_i \Gamma_u(r+i, m, x) \quad \textit{for } x \geq 0, \tag{11}$$

*(ii) the usual **slp** of X with priority $a \geq 0$ is given by*

$$\mathbb{E}\left[(X-a)_+\right] = \sum_{i=0}^{\infty} p_i \left[m(r+i)\Gamma_u(r+i+1, m, a) - a\Gamma_u(r+i, m, a)\right]. \tag{12}$$

Proof. If $f_X/f_\nu \in L^2(\nu)$ then Lemma 1 allows us to write f_X as in (9), and integrating this over $[x, \infty)$ yields the formula (11). Now consider the usual **slp** of X, and note that

$$\begin{aligned}\mathbb{E}\left[(X-a)_+\right] &= \int_a^{\infty} (x-a) f_X(x)\, \mathrm{d}x \\ &= \int_a^{\infty} x f_X(x)\, \mathrm{d}x - a\overline{F}_X(a). \end{aligned} \tag{13}$$

Then notice that for every $i \in \mathbb{N}_0$, we have that

$$\int_a^{\infty} x\, \gamma(r+i, m, x)\, \mathrm{d}x = \int_a^{\infty} x \frac{x^{r+i-1} \mathrm{e}^{-x/m}}{\Gamma(r+i) m^{r+i}}\, \mathrm{d}x$$

$$= m\frac{\Gamma(r+i+1)}{\Gamma(r+i)}\int_a^{\infty}\frac{x^{r+i}\mathrm{e}^{-x/m}}{\Gamma(r+i+1)m^{r+i+1}}\mathrm{d}x$$
$$= m(r+i)\Gamma_u(r+i+1,m,a). \quad (14)$$

Therefore substituting (9) and (11) into (13) and simplifying with (14) yields (12).

□

Let us make the connection between our approach and Erlang mixture more precise. Assuming that $f_X/f_\nu \in L^2(\nu)$ then taking the Laplace transform on both side of (9) yields

$$\mathcal{L}_X(s) = \sum_{i=0}^{\infty} p_i\left(\frac{1}{1+sm}\right)^{r+i} = \left(\frac{1}{1+sm}\right)^{r}\mathcal{P}\left(\frac{1}{1+sm}\right),$$

where $\mathcal{P}(z) = \sum_{i=1}^{\infty} p_i z^i$ denotes the generating function of the sequence of coefficient $\{p_i\ ,\ i \geq 1\}$. Now setting $z = \frac{1}{1+sm}$ allows to express the generating function $\mathcal{P}(z)$ in terms of the Laplace transform of X as

$$\mathcal{P}(z) = z^{-r}\mathcal{L}_X\left(\frac{1-z}{zm}\right).$$

Remark 3.2. *The approximation through an Erlang mixture consists in approximating the **pdf** of a nonnegative random variable X as*

$$f_X(x) = \sum_{i=1}^{\infty} p_i\gamma(i,m,x),\ \textit{for}\ x \geq 0.$$

*The function $\mathcal{P}(z)$ becomes then the probability generating function (**pgf**) of a counting random variable M, where $p_i = \mathbb{P}(M = i)$, for $i \geq 1$.*

The next example is designed to shed light on the link between our polynomial expansion and an Erlang mixture.

Example 1. *Suppose that we are interested in approximating the **pdf** of an exponential random variable* $\mathsf{Gamma}(1,\beta)$*. The generating function of the coefficients is then*

$$\mathcal{P}(z) = z^{1-r}\frac{m}{\beta+z(m-\beta)}.$$

*If one takes $r = 1$ and $m = \beta$ then $\mathcal{P}(z) = 1$ and the polynomial representation reduces to the exponential **pdf**. Choosing $0 < m < \beta$ leads to $\mathcal{P}(z) = \frac{m/\beta}{1-z(1-m/\beta)}$, which is the **pgf** of a geometric random variable; this recovers the fact that an exponential **rv** can be represented by a zero-truncated geometric sum of exponential **rv's**. For $m > \beta$, we have $\mathcal{P}(z) = \frac{m/\beta}{1+z(1-\beta/m)}$ which is an alternating sequence that decreases geometrically fast. Recall that our polynomial expansion is valid only if $m > \beta/2$, which means that when $\beta/2 < m \leq \beta$ our approach coincides with the Erlang mixture technique. It does not when $m > \beta$. When $m \leq \beta/2$, the Erlang mixture representation holds even though the integrability condition, which is a sufficient one, does not hold.*

The coefficients of the polynomials could be derived by differentiating the generating function $\mathcal{P}(z)$ as

$$p_i = \frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\mathcal{P}(z)\Big|_{z=0} = \text{Coefficient}(k, \text{MaclaurinSeries}(\mathcal{P}(z))),$$

for $i \geq 0$. In practice, the singularities of the function $\mathcal{P}(z)$ at zero mean this procedure is not viable. Instead, the p_i's are approximated by computing the q_k's and truncating their expression (10) up to a given order. The practical evaluation of the q_k's is discussed in Section 3.3.2.

A sufficient condition for $f_X/f_\nu \in L^2(\nu)$ is

$$f_X(x) = \begin{cases} \mathcal{O}(\mathrm{e}^{-x/\delta}) & \text{as } x \to \infty \text{ with } m > \delta/2, \\ \mathcal{O}(x^\beta) & \text{as } x \to 0 \text{ with } r < 2(\beta+1). \end{cases}$$

When X has a well-defined moment generating function one can typically choose r and m so this integrability condition is satisfied. Define the radius of convergence of the random variable X as

$$\rho_X = \sup\{s > 0\ ,\ \mathcal{L}\{f_X\}(-s) < +\infty\},$$

and consider the following result.

Proposition 2. *Let X be a non-negative random variable having a **pdf** f_X, having a well defined moment generating function, then*

$$f_X(x) = \mathcal{O}(\mathrm{e}^{-x\rho_X}) \text{ as } x \to \infty.$$

Proof. The result follows from applying Chernoff bound on the survival function, then De L'Hôpital's rule enables us to conclude. □

The integrability condition is satisfied if $m > \rho_X^{-1}/2$. When we consider heavy-tailed distributions, which is a desirable model characteristic in the applications, the integrability condition cannot be satisfied. The work-around is to use the expansion

$$\mathrm{e}^{-\theta x} f_X(x) = \sum_{k=0}^{\infty} q_k Q_k(x) f_\nu(x),$$

for some $\theta > 0$. Thus, we can use

$$f_X(x) = \mathrm{e}^{\theta x}\sum_{k=0}^{\infty} q_k Q_k(x) f_\nu(x) = \mathrm{e}^{\theta x}\sum_{i=0}^{\infty} p_i \gamma(r+i, m, x)$$

and since, when $1 - m\theta > 0$,

$$\mathrm{e}^{\theta x}\gamma(r+i, m, x) = (1-m\theta)^{-(r+i)}\gamma\Big(r+i, \frac{m}{1-m\theta}, x\Big)$$

we have

$$f_X(x) = \sum_{i=0}^{\infty} p_i (1-m\theta)^{-(r+i)}\gamma\Big(r+i, \frac{m}{1-m\theta}, x\Big) = \sum_{i=0}^{\infty}\widetilde{p}_i\gamma\Big(r+i, \widetilde{m}, x\Big),$$

where
$$\widetilde{p}_i = \frac{p_i}{(1-m\theta)^{r+i}} \quad \text{and} \quad \widetilde{m} = \frac{m}{1-m\theta}\,.$$
Calculating the q_i's and p_i's, topic covered in Section 3.3.2, requires a Laplace transform of $\mathrm{e}^{-\theta x} f_X(x)$ which is given by

$$\mathcal{L}\{\mathrm{e}^{-\theta x} f_X(x)\}(t) = \mathcal{L}\{f_X(x)\}(t+\theta).$$

The method described above is the same (up to some constants) as approximating the exponentially tilted distribution. This idea has been used in Asmussen **et al.** [3]. It is easily seen that taking $m > \theta^{-1}/2$ implies that $(\mathrm{e}^{-\theta x} f_X(x))/f_\nu(x) \in L^2(\nu)$.

3.3 Approximating densities of positive compound distributions

We now focus on variables S_N which admit a compound distribution. Since these distributions have an atom at 0, we put aside this singularity and focus on the defective **pdf** $f_{S_N}^+$. The discussion in Sections 3.1 and 3.2 also apply to defective densities. Namely, if $f_{S_N}^+/f_\nu \in L^2(\nu)$ then the expansion in Lemma 1 is valid, we have

$$f_{S_N}^+(x) = \sum_{k=0}^{\infty} q_k Q_k(x)\,\gamma(r,m,x) = \sum_{i=0}^{\infty} p_i \gamma(r+i,m,x), \text{ for } x>0,$$

where $q_k = \int_0^\infty Q_k(x) f_{S_N}^+(x)\,\mathrm{d}x$ and p_i is given by (10). Truncating the first summation yields
$$f_{S_N}^+(x) \approx \sum_{k=0}^{K} q_k Q_k(x)\,\gamma(r,m,x) = \sum_{i=0}^{K} \widehat{p}_i \gamma(r+i,m,x),$$
where $\widehat{p}_i = \sum_{k=i}^{K} q_k (-1)^{i+k}/[i!\,(k-i)!]\sqrt{k!\Gamma(k+r)/\Gamma(r)}$ for $i \le K$. The survival function $\overline{F}_{S_N}$ and the **slp** $\mathbb{E}\left[(S_N-a)_+\right]$ follows from Proposition 1. If the integrability condition is not satisfied then the exponentially tilted version of the defective **pdf** is expanded.

3.3.1 Choice of r and m

The parameters for the polynomial approximations are set differently for the light-tailed and heavy-tailed cases. In the light-tailed cases moment matching of order 2 is the natural procedure to set the values of r and m. We need to take into account the result in Proposition 2 and make sure that $m > \rho_X^{-1}/2$, where $\rho_X = \sup\{s>0\,;\,\mathcal{L}\{f_{S_N}^+\}(-s) < \infty\}$. Hence, the value of ρ_X depends on the distributions of N and U. The two distributions we use for modeling the claim frequency N are the *Poisson* and the *Pascal* distributions. The Poisson distribution is denoted by $\mathsf{Poisson}(\lambda)$ with **pmf**

$$f_N(k) = \frac{e^{-\lambda}\lambda^k}{k!}\,, \qquad \text{for } k = 0,1\ldots\,,$$

where $\lambda > 0$. We define the Pascal **rv** to be the number of failures counted before observing $\alpha \in \mathbb{N}_+$ successes, denoted $\mathsf{Pascal}(\alpha,p)$ with **pmf**

$$f_N(k) = \binom{\alpha+k-1}{k} p^\alpha q^k\,, \qquad \text{for } k = 0,1,\ldots\,.$$

Example 2. *Let N be Poisson distributed, the moment generating function of S_N is then given by*

$$\mathcal{L}_{S_N}(-s) = \exp\left[\lambda(\mathcal{L}_U(-s) - 1)\right].$$

The radius of convergence of S_N coincides with the one of U, $\rho_{S_N} = \rho_U$. In that case, we can set $r = 1$ and $m=\lambda\mathbb{E}(U)$ which corresponds to a moment matching procedure of order 1 or set $r = \lambda\mathbb{E}(U)^2/\mathbb{E}\left(U^2\right)$ and $m = \mathbb{E}\left(U^2\right)/\mathbb{E}(U)$ which, in turns, matches the two first moments.

Example 3. *Let N be Pascal distributed, the moment generating function of S_N is then given by*

$$\mathcal{L}_{S_N}(-s) = \left[\frac{p}{1 - q\mathcal{L}_U(-s)}\right]^{\alpha}.$$

The radius of convergence ρ_{S_N} is the positive solution of the equation $\mathcal{L}_U(-s) = q^{-1}$. We set $r = 1$ and $m = \rho_{S_N}^{-1}$.

The parametrization proposed in Example 3 is linked to the fact that it leads to the exact defective **pdf** in the case of a compound Pascal model with exponentially distributed claim sizes. First, we need to introduce the binomial distribution denoted by $\mathsf{Binomial}(n, p)$ with **pmf**

$$f_N(k) = \binom{n}{k} p^k q^{n-k}, \qquad \text{for } k = 0, 1, \ldots, n,$$

where $p \in (0, 1)$, $n \in \mathbb{N}_+$, and $p + q = 1$. The following lemma, adapted from [27], shows a useful correspondence between the Pascal and binomial distributions when used in compound sums with the exponential distribution.

Lemma 2. *Consider the random sums $X = \sum_{i=1}^{N_1} U_i$ and $Y = \sum_{i=1}^{N_2} V_i$, where*

$$N_1 \sim \mathsf{Pascal}(\alpha, p), \quad U_i \overset{\text{i.i.d.}}{\sim} \mathsf{Gamma}(1, \beta), \quad N_2 \sim \mathsf{Binomial}(\alpha, q), \quad V_i \overset{\text{i.i.d.}}{\sim} \mathsf{Gamma}(1, p^{-1}\beta),$$

where $p \in (0, 1)$, $\alpha \in \mathbb{N}_+$, $p + q = 1$, and where $\beta > 0$. Then we have $X \overset{\mathcal{D}}{=} Y$.

Proof. Both X and Y have the same Laplace transform, so $X \overset{\mathcal{D}}{=} Y$. □

Corollary 1. *Consider the compound sum $S_N = \sum_{i=1}^{N} U_i$ where $N \sim \mathsf{Pascal}(\alpha, p)$ and the $U_i \overset{\text{i.i.d.}}{\sim} \mathsf{Gamma}(1, \beta)$. Then the **sf** of S_N is given by*

$$\overline{F}_{S_N}(x) = \sum_{i=1}^{\alpha} \binom{\alpha}{i} q^i p^{\alpha-i}\, \Gamma_u\left(i, p^{-1}\beta, x\right),$$

*and its **slp** is given by*

$$\mathbb{E}\left[(S_N - a)_+\right] = \sum_{i=1}^{\alpha} \binom{\alpha}{i} q^i p^{\alpha-i} \left[\frac{i\beta}{p}\Gamma_u\left(i + 1, p^{-1}\beta, a\right) - a\Gamma_u\left(i, p^{-1}\beta, a\right)\right].$$

Proof. By Lemma 2 we treat S_N as if defined for $N \sim \mathsf{Binomial}(\alpha, q)$ and with $U_i \overset{\text{i.i.d.}}{\sim} \mathsf{Gamma}(1, p^{-1}\beta)$. The result follows by noting $S_n = U_1 + \cdots + U_n \sim \mathsf{Gamma}(n, p^{-1}\beta)$. □

One conclusion of Corollary 1 is that the exact solution coincides with our approximation when $r = 1$ and $m = p^{-1}\beta$ (and with $K \geq \alpha - 1$). Note that $p\beta^{-1}$ is the solution of the equation $\mathcal{L}_U(-s) = q^{-1}$ which is consistent with the parametrization proposed in Example 3.

In the heavy-tailed cases (**i.e.** when exponential tilting is required) we set $\theta = 1$, $m = \theta/2 = 1/2$ (at the lower limit for m; this gives $\widetilde{m} = 1$), and choose $r = \mathbb{E}[U]$.

3.3.2 Computation of the q_k's

The inherent challenge of the implementation of the polynomial method remains the evaluation of the coefficients $\{q_k, k \geq 0\}$. Recall that

$$q_k = \int_0^\infty Q_k(x) f_{S_N}^+(x)\,\mathrm{d}x,\ \ k \geq 0.$$

We propose an evaluation based on the Laplace transform $\mathcal{L}\{f_{S_N}^+\}$. Define the generating function of the sequence $\{q_k c_k, k \geq 0\}$ as $\mathcal{Q}(z) = \sum_{k=0}^\infty q_k c_k z^k$, where

$$c_k = \left(\frac{\Gamma(k+r)}{\Gamma(k+1)\Gamma(r)}\right)^{1/2}, \quad \text{for } k \geq 0\,.$$

The following result establishes a link between the Laplace transform of $f_{S_N}^+$ and the generating function $\mathcal{Q}(z)$.

Proposition 3. *Assume that $f_{S_N}^+/f_\nu \in L^2(\nu)$, then we have*

$$\mathcal{Q}(z) = (1+z)^{-r}\mathcal{L}\{f_{S_N}^+\}\left[\frac{-z}{m(1+z)}\right]. \tag{15}$$

Proof. As $f_{S_N}^+/f_\nu \in L^2(\nu)$, the polynomial representation of $f_{S_N}^+$ follows from the application of Lemma 1 with

$$f_{S_N}^+(x) = \sum_{k=0}^\infty \sum_{i=0}^k q_k \frac{(-1)^{i+k}}{i!\,(k-i)!}\sqrt{\frac{k!\Gamma(k+r)}{\Gamma(r)}}\gamma(r+i, m, x). \tag{16}$$

Taking the Laplace transform in (16) yields

$$\begin{aligned}
\mathcal{L}\{f_{S_N}^+\}(s) &= \left(\frac{1}{1+sm}\right)^r \sum_{k=0}^\infty q_k \sum_{i=0}^k (-1)^{k+i}\left(\frac{\Gamma(k+r)}{\Gamma(k+1)\Gamma(r)}\right)^{1/2}\binom{k}{i}\left(\frac{1}{1+sm}\right)^i \\
&= \left(\frac{1}{1+sm}\right)^r \sum_{k=0}^\infty q_k c_k (-1)^k \sum_{i=0}^k \binom{k}{i}\left(\frac{-1}{1+sm}\right)^i \\
&= \left(\frac{1}{1+sm}\right)^r \sum_{k=0}^\infty q_k c_k (-1)^k \left(\frac{sm}{1+sm}\right)^k \\
&= \left(1 - \frac{sm}{1+sm}\right)^r \mathcal{Q}\left(-\frac{sm}{1+sm}\right).
\end{aligned}$$

Thus (15) follows from letting $z = -sm/(1+sm)$. □

The Laplace transform of the defective **pdf** $f^{+}_{S_N}$ is given by

$$\mathcal{L}\{f^{+}_{S_N}\}(s) = \mathcal{L}_{S_N}(s) - \mathbb{P}(N=0).$$

The coefficients of the polynomials can be derived after differentiation of the generating function $\mathcal{Q}(z)$ as

$$q_k = \frac{1}{c_k}\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z^k}\mathcal{Q}(z)\Big|_{z=0} = \frac{1}{c_k}\mathrm{Coefficient}(k, \mathrm{MaclaurinSeries}(\mathcal{Q}(z))).$$

4 Laplace transform inversion approximations

We present in this section a method inspired from the work of Abate and Whitt [1] to recover the survival function of a compound distribution from the knowledge of its Laplace transform. The methodology is further applied to the computation of **slp's** by taking advantage of the connection between the **slp** of S_N and the survival function of the equilibrium distribution of S_N. We begin by stating some useful transform relations, then discuss the general Laplace inversion framework that we will use, and will apply the method to the compound distribution problem.

4.1 Numerical Laplace inversion

A function f can be recovered from its Laplace transform by a standard Bromwich integral. We assume $f : \mathbb{R}_+ \to \mathbb{R}_+$, is a measurable function with locally bounded variation. To define the Bromwich integral, first select a $\gamma > 0$ (we discuss this choice later), then

$$f(x) = \frac{2\mathrm{e}^{\gamma x}}{\pi}\int_0^{\infty} \cos(xs)\Re\left[\mathcal{L}\{f\}(\gamma + is)\right]\,\mathrm{d}s.$$

We apply a basic numerical integration system to this integral by first *discretizing* the integral and then *truncating* the resulting infinite sum. In both steps, we follow the steps of Abate and Whitt [1].

4.1.1 Discretization

We will use a semi-infinite trapezoidal rule, despite the apparent simplicity of the method. With a grid size $h > 0$, this discretization yields

$$f(x) \approx f_{\mathrm{disc}}(x) \equiv \frac{2\mathrm{e}^{\gamma x}}{\pi}\cdot h\left\{\frac{1}{2}\mathcal{L}\{f\}(\gamma) + \sum_{j=1}^{\infty}\cos(x\cdot hj)\Re\left[\mathcal{L}\{f\}(\gamma + i\cdot hj)\right]\right\},$$

since $\Re\left[\mathcal{L}\{f\}(\gamma)\right] = \mathcal{L}\{f\}(\gamma)$. We simplify this by choosing $h = \pi/(2x)$ and $\gamma = a/(2x)$ for an $a > 0$, achieving

$$f_{\mathrm{disc}}(x) = \frac{\mathrm{e}^{a/2}}{2x}\mathcal{L}\{f\}\left(\frac{a}{2x}\right) + \frac{\mathrm{e}^{a/2}}{x}\sum_{k=1}^{\infty}(-1)^k\Re\left[\mathcal{L}\{f\}\left(\frac{a + i\cdot 2\pi k}{2x}\right)\right]. \tag{17}$$

From Theorem 5.5.1 of [30] we have that the *discretization error* (also called *sampling error*) is simply

$$f_{\text{disc}}(x) - f(x) = \sum_{k=1}^{\infty} \mathrm{e}^{-ak} f\Big((2k+1)x\Big) . \tag{18}$$

In particular, if $0 \le f(x) \le 1$, then

$$f_{\text{disc}}(x) - f(x) \le \frac{\mathrm{e}^{-a}}{1-\mathrm{e}^{-a}} . \tag{19}$$

There are no absolute value signs here — the discretization introduces a systematic overestimate of the true function value. Also, (18) implies a should be as large as possible (limited eventually by finite-precision computation). The benefit of knowing this result is slightly offset by the requirement that h and γ now be functions of x rather than constants.

4.1.2 Truncation

Due to the infinite series, the expression in (17) cannot be directly computed, thus it has to be truncated. The arbitrary-seeming choice of h and γ in Section 4.1.1 not only allows for calculation of the discretization error, but also benefits the truncation step. This is because the sum in (17) is (nearly) of alternating sign, and thus *Euler series acceleration* can be applied to decrease the truncation error. Define for $\ell = 1, 2, \ldots$

$$s_\ell(x) \equiv \frac{\mathrm{e}^{a/2}}{2x}\mathcal{L}\{f\}\left(\frac{a}{2x}\right) + \frac{\mathrm{e}^{a/2}}{x}\sum_{k=1}^{\ell}(-1)^k \Re\left[\mathcal{L}\{f\}\left(\frac{a+i\cdot 2\pi k}{2x}\right)\right] .$$

Then, for some positive integers M_1 and M_2,

$$f(x) \approx f_{\text{disc}}(x) \approx f_{\text{approx}}(x) \equiv \sum_{k=0}^{M_1}\binom{M_1}{k} 2^{-M_1} s_{M_2+k}(x) . \tag{20}$$

4.2 Estimators of survival function and stop-loss premium for compound distributions

For a random sum S_N, we consider using the technique above to evaluate the **sf** $\overline{F}_{S_N}$ and the **slp's** from their Laplace transform. We invert $\mathcal{L}\{\overline{F}_{S_N}\}$, but note that inverting $\mathcal{L}\{F_{S_N}\}$ produces almost identical results.

This inversion easily gives estimates of $\overline{F}_{S_N}$, though evaluating the **slp's** requires extra thought. As noted in Dufresne **et al.** [8], we have that

$$\mathbb{E}\left[(S_N - d)_+\right] = \mathbb{E}(S_N)\overline{F_{S_N^*}}(d), \tag{21}$$

where S_N^* is a random variable under the *equilibrium distribution* with density

$$f_{S_N^*}(x) = \begin{cases} \overline{F}_{S_N}(x)/\mathbb{E}(S_N), & \text{for } x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

and Laplace transform

$$\mathcal{L}_{S_N^*}(s) = \frac{1 - \mathcal{L}_{S_N}(s)}{s\mathbb{E}(S_N)}.$$

The **slp** is then obtained, replacing in (21) the **sf** of S_N^* by its approximation through (20).

5 Numerical illustrations

In this section, we illustrate the performance of the two proposed numerical procedures. Section 5.1 focuses on approximating the **sf** and the **slp** associated to aggregated claim sizes, while Section 5.2 considers the approximation of the finite-time ruin probability with no initial reserves using formula (5).

For each test case we compare the orthogonal polynomial approximation, the Laplace inversion approximation, and for the crude Monte Carlo approximation. For the cases when U is gamma distributed, we use the fact that S_n is Erlang distributed to produce an approximate distribution for S_N by truncating N to be less than some large level.

The parameters for the polynomial approximations has been discussed in Section 3.3.1, the calibration is depending on the assumptions over the claim frequency and claim sizes distribution. The parameters for the Laplace inversion technique are set to $M_1 = 11$, $M_2 = 15$ and $a = 18.5$ following the example of Rolski **et al.** [30, Chapter 5, Section 5]; note, this choice of a implies that the discretization error is less than 10^{-8}, as derived from (19).

In each plot, the first subplot shows the estimates each estimator produces, and the second shows the *approximate absolute error*. We define this, for estimator $i \in \{1, \dots, I\}$, as

$$\begin{aligned}\text{ApproximateAbsoluteError}(\widehat{f}_i, x) &:= \widehat{f}_i(x) - \text{Median}\Big\{\widehat{f}_1(x), \dots, \widehat{f}_I(x)\Big\} \\ &\approx \widehat{f}_i(x) - f(x) =: \text{AbsoluteError}(\widehat{f}_i, x)\,.\end{aligned}$$

When the different estimators cross each other, the median obtains an unrealistically jagged character. We therefore use as reference a slightly smoothed version of the median, achieved in Mathematica using `GaussianFilter[Medians, 2]`. As noted earlier, all of the code used is available online [11].

5.1 Survival function and stop-loss premium computations

To ensure both estimators were implemented correctly, we applied the estimators to the case where $N \sim \mathsf{Pascal}(\alpha = 10, p = 3/4)$ and $U \sim \mathsf{Gamma}(r = 1, m = 1/6)$. Corollary 1 tells us the orthogonal approximation (with $r = 1$, $m = \lambda/p = 2/9$ and $K = \alpha - 1 = 9$) is equivalent to the true function, which we verified, and the Laplace inversion errors in Tables 1 and 2 are acceptably small.

Table 1: Relative errors for the Laplace inversion **sf** estimator

x	0.5	1	1.5	2	2.5
Error	7.27e-7	1.92e-6	5.86e-6	1.78e-5	4.01e-5

Table 2: Relative errors for the Laplace inversion **slp** estimator

a	0.5	1	1.5	2	2.5
Error	8.68e-7	2.27e-6	5.92e-6	1.12e-5	-2.12e-5

Test 1. $N \sim \mathsf{Poisson}(\lambda = 2)$, *and* $U \sim \mathsf{Gamma}(r = 3/2, m = 1/3)$

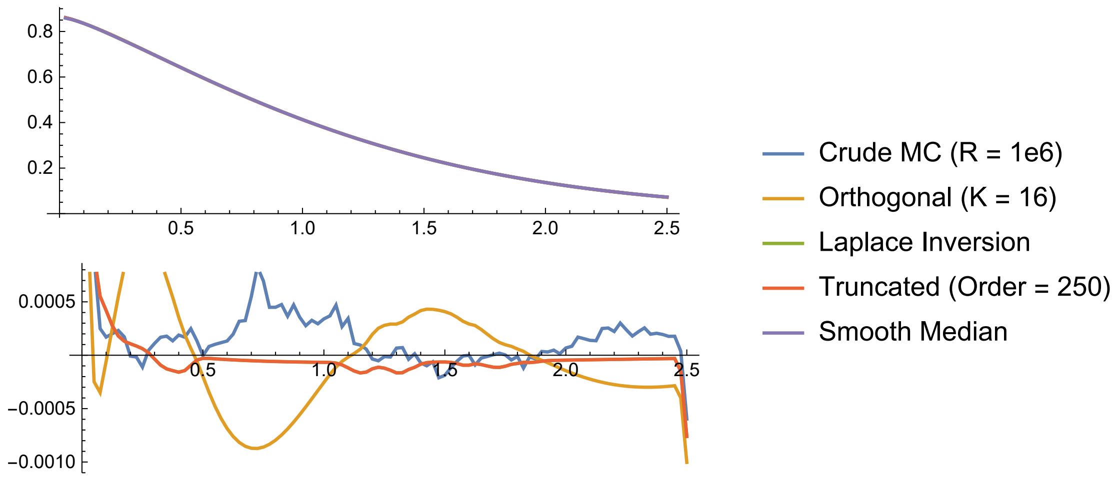

Figure 1: Survival function estimates and approximate absolute error.

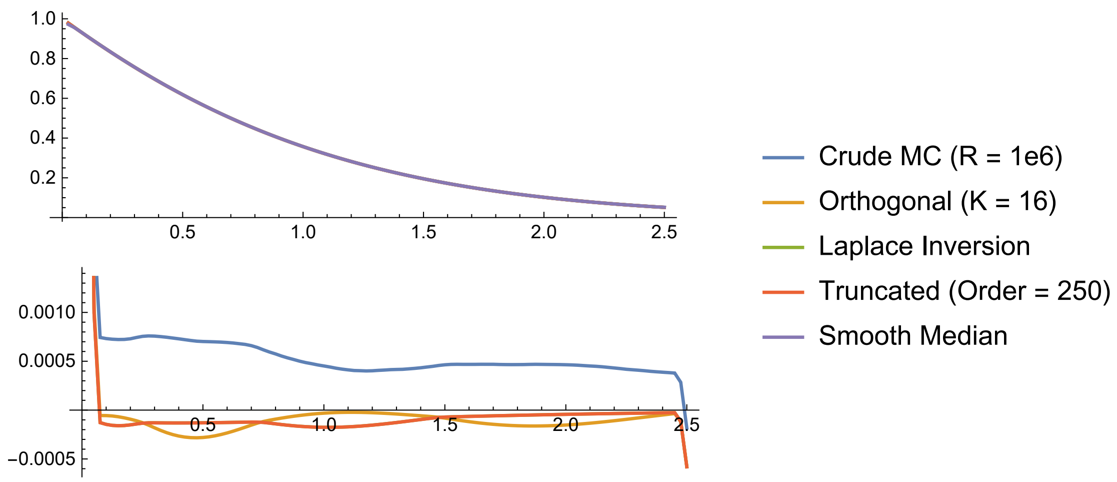

Figure 2: Stop-loss premium estimates and approximate absolute error.

Test 2. $N \sim \mathsf{Pascal}(\alpha = 10, p = 1/6)$, *and* $U \sim \mathsf{Gamma}(r = 3/2, m = 1/75)$

This test case (up to the scaling constant) has been considered by Jin **et al.** [17, Example 3]. In the plots for this test case, the orthogonal estimator, the Laplace inversion estimator, and the truncated estimator all give the same values and hence are hidden underneath the red line for the truncated estimator.

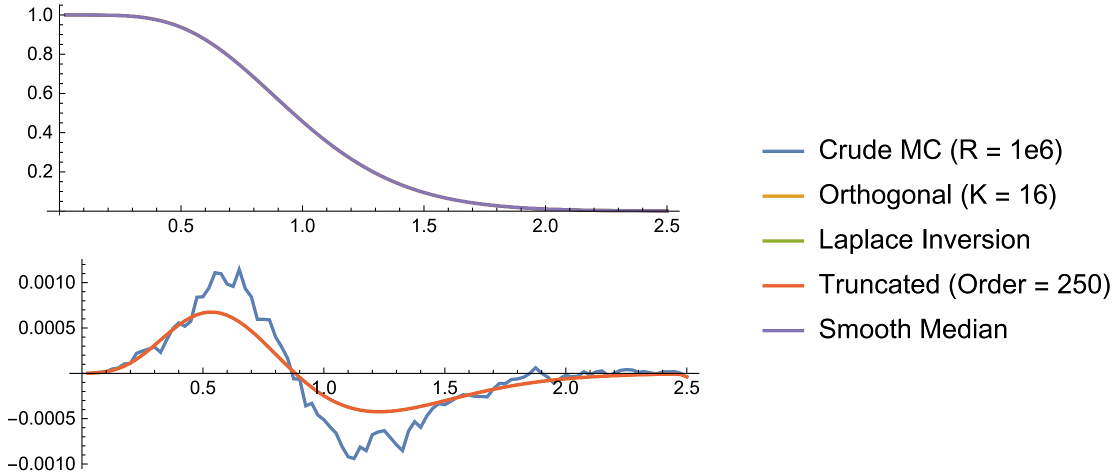

Figure 3: Survival function estimates and approximate absolute error.

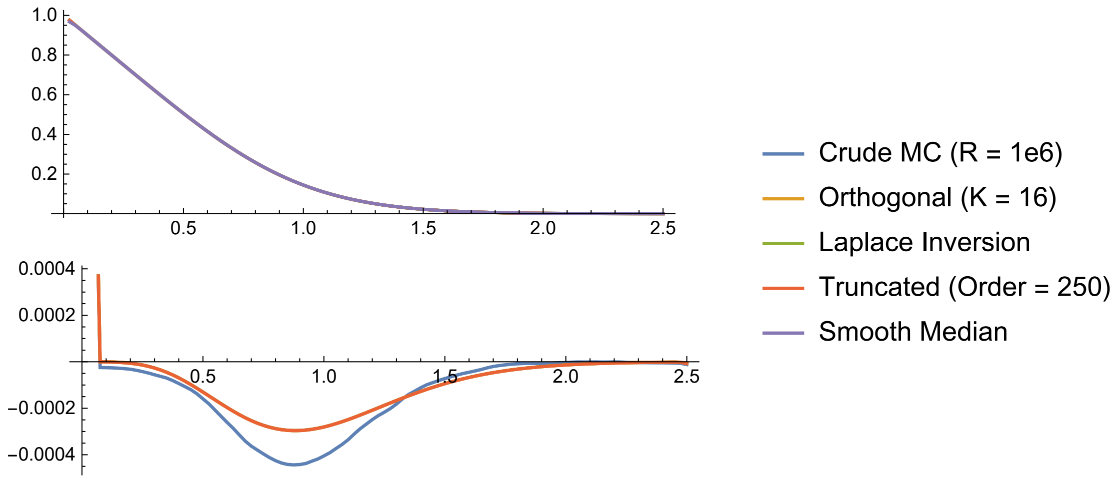

Figure 4: Stop-loss premium estimates and approximate absolute error.

Test 3. *$N \sim$ Poisson$(\lambda = 4)$, and $U \sim$ Pareto$(a = 5, b = 11, \theta = 0)$*

The survival function for U, given $x \geq \theta = 0$, is

$$\overline{F}_U(x) = \left(1 + \frac{x - \theta}{a}\right)^{-b} = \left(1 + \frac{x}{5}\right)^{-11} .$$

We note that the Laplace inversion estimator breaks down for small values of x or a in this test case. The specific error given is an "out of memory" exception when Mathematica is attempting to do some algebra with extremely large numbers. It is unclear whether a different implementation or selection of parameters would fix this behaviour.

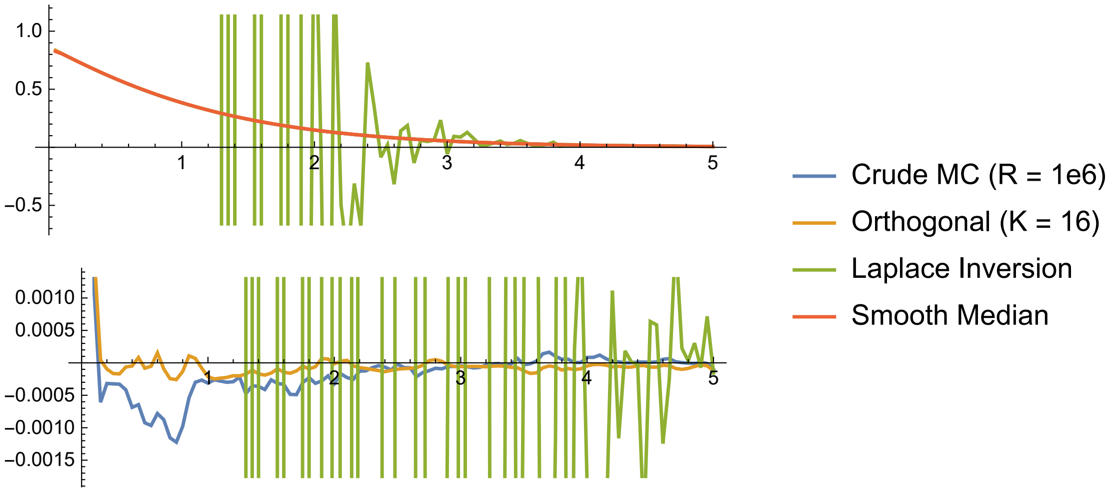

Figure 5: Test 3: Survival function estimates and approximate absolute error.

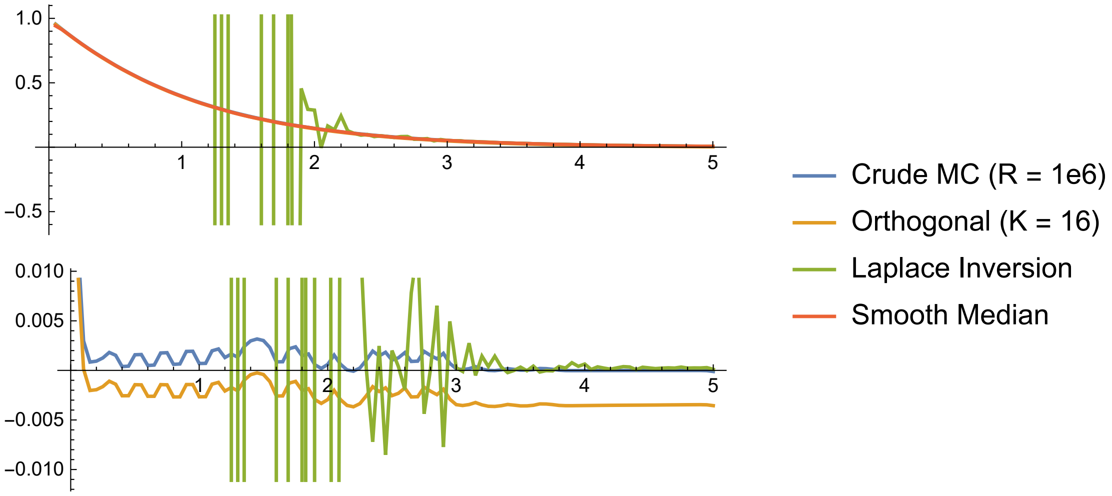

Figure 6: Test 3: Stop-loss premium estimates and approximate absolute error.

Test 4. $N \sim \mathsf{Pascal}(\alpha = 2, p = 1/4)$, *and* $U \sim \mathsf{Weibull}(\beta = 1/2, \lambda = 1/2)$

The survival function for U, given $x \geq 0$, is

$$\overline{F}_U(x) = \exp\left\{-\left(\frac{x}{\lambda}\right)^{\beta}\right\} = \exp\left\{-\sqrt{2x}\right\} .$$

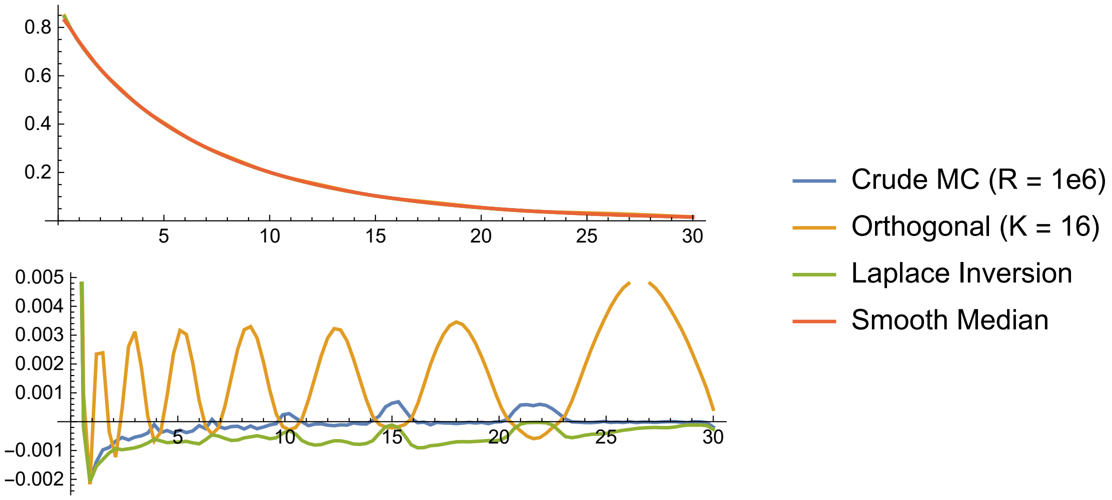

Figure 7: Test 4: Survival function estimates and approximate absolute error.

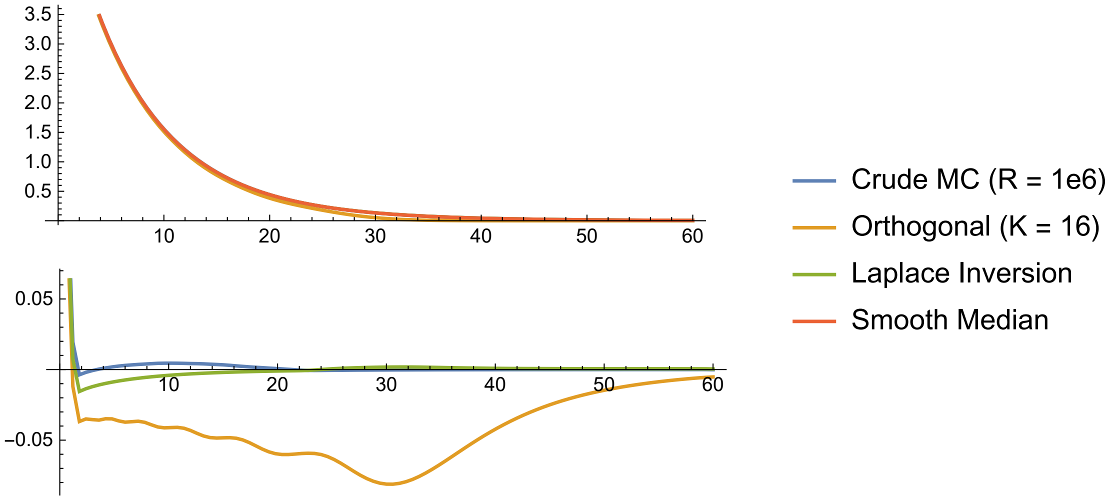

Figure 8: Test 4: Stop-loss premium estimates and approximate absolute error.

5.2 Finite-time ruin probability with no initial reserve

In this paper we have used common random numbers for all crude Monte Carlo estimators to smooth their estimates. However in the case of ruin probabilities, the distribution from which we are simulating Poisson(λt) is changing for each point, so the technique cannot be applied in the traditional way. Thus the crude Monte Carlo estimates in following plots are not as smooth as above.

Test 5. $\lambda = 4$ *and* $U \sim \mathsf{Gamma}(r = 2, m = 2)$ *and* $c = 1$

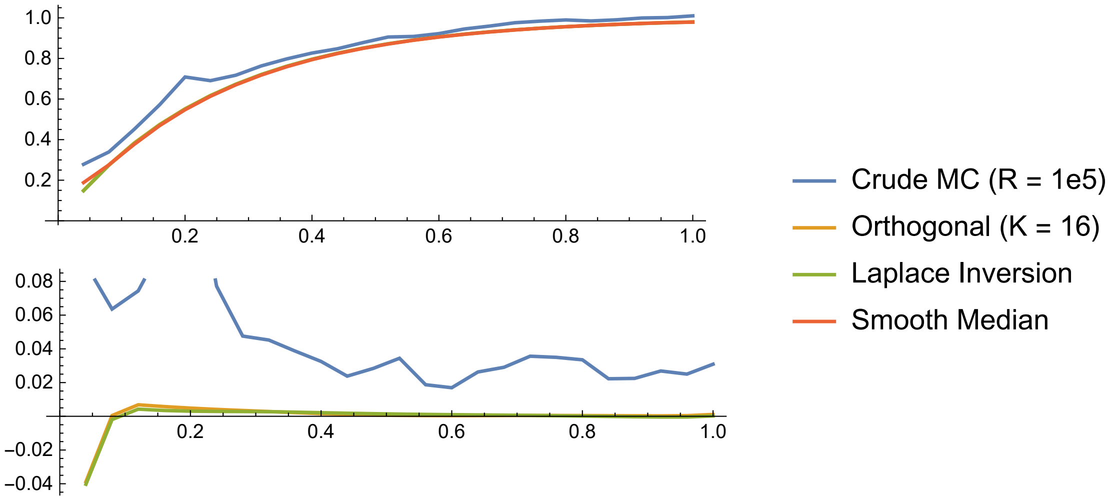

Figure 9: Test 5: Ruin probability $\psi(0, t)$ estimates and approximate absolute error.

Test 6. $\lambda = 2$ *and* $U \sim \text{Pareto}(a = 5, b = 11, \theta = 0)$ *and* $c = 1$

See the discussion of Test 3 for a description of the Laplace inversion estimator's poor behaviour when Pareto variables are involved.

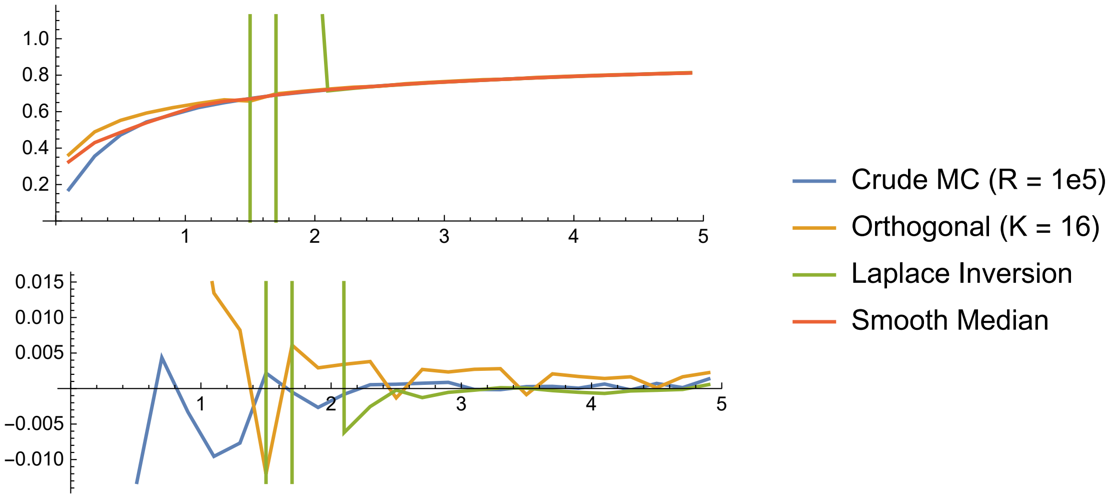

Figure 10: Test 6: Ruin probability $\psi(0, t)$ estimates and approximate absolute error.

5.3 Concluding remarks

The orthogonal polynomial method has performed well across all the test cases studied. The accuracy is acceptable even with a rather small order of truncation $K = 16$. It produces an approximation having an analytical expression, which is desirable, and in a timely manner. The precision may be improved by adding more terms in the expansions. The main drawback is probably the need for a parametrization tailored to the case studied.

The Laplace transform inversion method yields outstanding result in terms of accuracy. It failed to provide a stable approximation for Pareto distributed claim sizes. The parametrization is automatic and seems to fit the different case studied (except the Pareto one).

Both of the methods are easy to implement and beat a simple truncation or a crude Monte-Carlo approach, which is the main conclusion of our work.

Acknowledgments

Pierre-Olivier Goffard was partially funded by a Center of Actuarial Excellence educational grant given to the University of California, Santa Barbara, by the Society of Actuaries. Patrick J. Laub was supported by an Australian Government Research Training Program Scholarship and an Australian Research Council Centre of Excellence for Mathematical & Statistical Frontiers Scholarship.

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