Duality in ruin problems for ordered risk models

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Abstract

On one hand, an ordered dual risk model is considered where the profit arrivals are governed by an order statistic point process (OSPP). First, the ruin time distribution is obtained in terms of Abel-Gontcharov polynomials. Then, by duality, the ruin time distribution is deduced for an insurance model where the claim amounts correspond to the inter-arrival times in an OSPP. On the other hand, an ordered insurance model is considered with an OSPP as claim arrival process. Lefèvre and Picard [28] determined the finite-time ruin probability in terms of Appell polynomials. Duality is used to derive the ruin probability in a dual model where the profit sizes correspond to the inter-arrival times of an OSPP.

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1 Introduction

Dual risk models describe the wealth of a company for which the operational cost is deterministic and the profits occur stochastically. Their appellation comes from the duality with insurance (or primal) risk models for which the premium income is deterministic and the claims arrive stochastically. There is an extensive literature on insurance risk models. Dual risk models have received an increasing interest in recent years.

Dual risk model. A company holds an initial capital $v > 0$ and faces running costs at a constant rate $a > 0$. The company makes profits over time that form a sequence $\{Y_i, i \geq 0\}$ of i.i.d. non-negative random variables. These profits occur according to a counting process $\{M(t), t \geq 0\}$, independently of the Y_i. The associated wealth process $\{W(t), t \geq 0\}$ is given by

$$W(t) = v - at + \sum_{i=1}^{M(t)} Y_i, \quad t \geq 0. \tag{1}$$

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The ruin time, σ_v, is the first instant at which the wealth process falls at the level 0, i.e.

$$\sigma_v = \inf\{t \geq 0 : W(t) = (\leq) \, 0\}. \tag{2}$$

The dual risk model is considered e.g. in the books by Cramér [12], Seal [37], Takàcs [41], Grandell [24] and Asmussen and Albrecher [3]. The model is suitable for risky business sectors such as oil prospection, pharmaceutical research or new technology development. So, Bayraktar and Egami [6] use it to describe the financial reserves of venture capital funds, and Bertail et al. [8] to model the exposure to a given food contaminant. Another application is in life insurance when a company pays annuities on a regular basis and receives a part of the reserves at each policyholder death.

Many of the works on dual models focus on optimal dividend problems. We refer e.g. to Avanzi et al. [4], Gerber and Smith [21], Albrecher et al. [2], Dai et al. [13], Cheung [11], Afonso et al. [1], and Bergel et al. [7]. First-passage problems and ruin time are also much studied; see e.g. Landriault and Sendova [26], Mazza and Rullière [31], Zhu and Yang [43], Yang and Sendova [42] and Dimitrova et al. [18]. There exists here a close and important connection with queueing models; see e.g. Frostig [19], Badescu et al. [5] and Frostig and Keren-Pinhasik [20].

The classical dual model is the compound Poisson case where the counting process $\{M(t), \, t \geq 0\}$ in (1) is a Poisson process. The Sparre-Andersen case where $\{M(t), \, t \geq 0\}$ is a renewal process and other extensions like the Markov-modulated case have been investigated to a certain extent.

In this paper, we first examine a dual risk model where $\{M(t), \, t \geq 0\}$ is an order statistic point process (OSPP). Such a model is named ordered dual in the sequel. The class class of OSPP was characterized by Puri [34], further to earlier partial results. The key property of an OSPP is that conditionally on the number of profit arrivals up to time $t \geq 0$, the jump times are distributed as the order statistics for a random sample drawn from some continuous distribution with support $(0, t)$. The OSPP cover the (mixed) Poisson process, the linear birth process with immigration and the linear death counting process. This class of counting processes was proposed to model claim frequencies in insurance by Lefèvre and Picard [28], after the pioneering works of De Vylder and Goovaerts [14, 15]. It was used later by Sendova and Zitikis [38], Lefèvre and Picard [29, 30] and Dimitrova et al. [17].

For that dual model, our purpose is to derive an explicit formula for the distribution of the ruin time σ_v. To this end, we use the representation of the joint distribution of the order statistics from a uniform distribution through the so-called Abel-Gontcharov polynomials. This family of polynomials is little known and related to the more standard Appell polynomials. A review on both polynomial families is provided in Lefèvre and Picard [30], with applications in risk modelling.

Insurance risk model. An insurance company has an initial capital $u \geq 0$ and receives premiums at a constant rate $c > 0$. The company covers claim amounts over time that form a sequence $\{X_i, \, i \geq 0\}$ of i.i.d. non-negative random variables. These claims occur according to a counting process $\{N(t), \, t \geq 0\}$, independently of the X_i. The associated reserve process $\{R(t), \, t \geq 0\}$ is given by

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad \geq 0. \tag{3}$$

The ruin time, τ_u, is the first instant at which the reserve process becomes negative, i.e.

$$\tau_u = \inf\{t \geq 0 : R(t) < 0\}. \tag{4}$$

Much research is devoted to insurance risk models, especially for ruin related problems. The reader is referred e.g. to the books of Asmussen and Albrecher [3] and Dickson [16]. The traditional case is the compound Poisson model where the counting process $\{N(t),\, t \geq 0\}$ in (3) is a Poisson process. The Sparre-Andersen model supposes that $\{N(t),\, t \geq 0\}$ is a renewal process. A number of generalizations and variants of these models have been considered. This is the case e.g. of the mixed Poisson model (see Grandell [25]).

It is well-known that to the insurance risk model (3) is associated a dual risk model (1) whose characteristics are inverted in a precise sense. Specifically, the profits in the dual model correspond to the inter-arrival times in the insurance model while the inter-arrival times in the dual model correspond to the claim sizes in the insurance model, and the cost rate in the dual model is the inverse of the premium rate in the insurance model. This duality is pointed out and exploited in various works in insurance. Let us mention e.g. the recent papers by Shi and Landriault [39], Mazza and Rullière [31], Borovkov and Dickson [10] and Dimitrova et al. [18]. A similar duality property is used with Lévy processes in finance.

This link between the primal and dual models can provide a simple approach for tackling ruin problems. Mazza and Rullière [31] start with the compound Poisson dual model, a recursive formula for the finite-time ruin probability is derived (similar to existing recursive formula for the finite-time ruin probability in the insurance risk model, see e.g. Picard and Lefèvre [33], Loisel and Rullière [36] and Lefèvre and Loisel [27]) and then pass to the insurance model with exponential claim amounts. Dimitrova et al. [18] apply results for the insurance model to obtain the ruin probability in the corresponding dual model.

In the same vein, we will make here a round trip between the dual and insurance models. As announced above, we first derive a formula for the distribution of the ruin time σ_v in an ordered dual risk model. From this formula, we then deduce the distribution of the ruin time τ_u in the Sparre-Andersen insurance model where the claim arrivals are governed by a renewal process and the claim amounts are distributed as the inter-arrival times in an OSPP. As a special case, we recover a result obtained by Borovkov and Dickson [10] for the case of i.i.d. exponential claim amounts.

Our second journey is from the primal to the dual and concerns now the finite-time ruin probability. We start with an ordered insurance model where the claim arrivals are described by an OSPP. For this model, Lefèvre and Picard [28, 29] derived a formula for the ruin probability in terms now of Appell polynomials. By duality, we can then obtain the finite-time ruin probability in the associated dual risk model where the profit arrivals are governed by a renewal process and the profit sizes are distributed as the inter-arrival times in an OSPP.

It is worth underlining that the approach by duality enables us to deal with ruin problems in primal or dual risk models of renewal type which allow for some dependence between the claim or profit sizes. Models with dependent claims or profits are usually difficult to study. The present method, rather simple, relies on a preliminary study of the ruin in the associated dual and primal models. Such a study is possible when the latter models are ordered, i.e. the profits or claims arrive according to an OSPP.

Summary. The paper is organized as follows. Section 2 gives an overview of the order statistic point processes. In Section 3, we obtain the ruin time distribution in the ordered dual model when the profits arrivals are governed by an OSPP. In Section 4, we deduce the ruin time distribution in a Sparre-Andersen insurance model where the claim amounts correspond to the inter-arrival times of an OSPP. In Section 5, we obtain the finite-time ruin probability in the associated dual model where the profit sizes correspond to the inter-arrival times of an OSPP.

2 Order statistic property

The Poisson process is the traditional model for counting events that arise randomly in the course of time. Of simple construction, it has also many desirable properties. In particular, it belongs to the class of order statistic point processes.

Definition 2.1. *A point process $\{N(t),\, t \geq 0\}$, with $N(0) = 0$, is an OSPP if for every $n \geq 1$, provided $\mathbb{P}[N(t) = n] > 0$, then conditioned upon $[N(t) = n]$, the successive jump times $(T_1, T_2, \ldots, T_n)$ are distributed as the order statistics $[U_{1:n}(t), \ldots, U_{n:n}(t)]$ of a sample of n i.i.d. random variables on $[0, t]$, distributed as a variable $\mathcal{U}(t)$ of distribution function $\mathbb{P}[\mathcal{U}(t) \leq s] = F_t(s)$, $0 \leq s \leq t$.*

De Vylder and Goovaerts [14, 15] introduced a risk model, named homogeneous, that generalizes the classical Cramér-Lundberg risk model. When defined on an infinite horizon, this model supposes that the claim arrival process satisfies the above order statistic property where $\mathcal{U}(t)$, $t > 0$, is uniform on $(0, t)$ (as it is for the Poisson process). Their research was made independently of the existing literature on OSPP.

More recently, Lefèvre and Picard [28] introduced an insurance risk model in which claim arrivals are described by an OSPP. This paper was continued in Lefèvre and Picard [29, 30]. The purpose there is to determine the finite-time ruin probability in such a model. We also mention the related works by Sendova and Zitikis [38], Dimitrova et al. [17] and Goffard [22]. Outside of the insurance context, Goffard and Lefèvre [23] studied the first-crossing problem of an OSSP through general boundaries.

A complete representation of the class of OSPP was obtained by Puri [34], following on earlier works. A key result is recalled below.

Proposition 2.2. *(Puri [34])*
Let $\{N(t),\, t \geq 0\}$ be an OSPP where $\mu(t) = \mathbb{E}[N(t)]$ is finite for all t.

(i) If $\lim_{t\to\infty} \mu(t) = \infty$, then $\{N(t),\, t \geq 0\}$ is a mixed Poisson process up to a time-scale transformation. So, it can be represented as

$$N(t) = \mathcal{P}\left[D\nu(t)\right], \quad t \geq 0, \quad a.s., \tag{5}$$

where $\mathcal{P}(\theta)$ denotes a Poisson distribution of parameter θ, D is an independent non-negative random variable and $\nu(t)$ is a deterministic time function.

(ii) If $\lim_{t\to\infty} \mu(t) = \mu < \infty$, then $\{N(t),\, t \geq 0\}$ is a death counting process in which the individual lifetimes are i.i.d. random variables of distribution function $\mu(t)/\mu$, $t \geq 0$, and there is initially an independent random number Z of individuals in the population. So, it can be represented as a mixed binomial process, i.e.

$$N(t) = \mathcal{B}\left[Z, \mu(t)/\mu\right], \quad t \geq 0, \quad a.s., \tag{6}$$

where $\mathcal{B}(z,p)$ denotes a binomial distribution of parameters z and p.

For both cases, the order statistic property holds with

$$F_t(s) = \mu(s)/\mu(t), \quad 0 \leq s \leq t. \tag{7}$$

Here are some simple special cases used in various applications.

Particular OSPP

(1) A Poisson process of parameter λ. Here, $N(t)$ has a Poisson distribution of mean $\mu(t) = \lambda t$. So, (5) holds with $D = \lambda$ a.s. and $\nu(t) = t$, and (7) gives $F_t(s) = s/t$.

(2) An inhomogeneous Poisson process of continuous intensity function $\lambda(t)$. Here, $N(t)$ has a Poisson distribution of mean $\mu(t) = \int_0^t \lambda(z)dz$. So, (5) holds with $D = 1$ a.s. and $\nu(t) = \mu(t)$, and $F_t(s)$ is given by (7).

(3) A mixed Poisson process of mixing variable Λ. Here, $N(t)$ has a mixed Poisson distribution of random parameter Λ, with mean $\mu(t) = E(\Lambda)t$. So, (5) holds with $D = \Lambda$ and $\nu(t) = t$, and (7) gives $F_t(s) = s/t$ (independently of Λ).

For instance, if Λ has a gamma distribution $\Gamma(\gamma, \beta)$, then the mixed Poisson process is a negative binomial process (or Pólya process) of parameters γ and β. So, $N(t)$ has a negative binomial distribution:

$$\mathbb{P}\left[N(t) = n\right] = \binom{\gamma + n - 1}{n} \left(\frac{t}{t+\beta}\right)^n \left(\frac{\beta}{t+\beta}\right)^\gamma, \quad n \geq 0,$$

with mean $\mu(t) = (\gamma/\beta)t$.

(4) A linear birth process of rate α and with immigration of rate λ. Here, $N(t)$ has a negative binomial distribution:

$$\mathbb{P}\left[N(t) = n\right] = \binom{\lambda/\alpha + n - 1}{n} (1 - e^{-\alpha t})^n \, e^{-\lambda t}, \quad n \geq 0,$$

with mean $\mu(t) = (\lambda/\alpha)(e^{\lambda t} - 1)$.

This process can also be considered as a non-homogeneous mixed Poisson process (5) for which D has a gamma distribution $\Gamma(\lambda/\alpha, 1)$ and $\nu(t) = \exp(\lambda t) - 1$.

(5) A linear death counting process of rate α and initial size n. Here, $N(t)$ has a binomial distribution:

$$\mathbb{P}\left[N(t)=n\right]=\binom{z}{n}(1-e^{-\alpha t})^{n}e^{-\alpha t(z-n)},\quad 0\leq n\leq z,$$

with mean $\mu(t)=z(1-e^{-\alpha t})$, of finite limit $\mu=z$ as $t\to\infty$. So, (6) holds with $Z=z$ a.s. and the lifetimes are i.i.d. exponentials of parameter α.

In the sequel, we will be also interested in the distributions of the inter-arrival times Δ_i^T, $i\geq 1$, in an OSPP; so, $\Delta_i^T=T_i-T_{i-1}$with $T_0=0$. They are named here level spacings (as a level spacing in physics is the difference between two consecutive elements). There is no general formula for the joint disributions of the Δ_i^T, unfortunately. Nevertheless, a simple expression is available for the previous special cases.

Particular level spacings

(1) For a Poisson process, the Δ_i^T are i.i.d. exponentials of parameter λ.

(2) For an inhomogeneous Poisson process, the Δ_i^T are dependent and each vector $(\Delta_1^T,\dots,\Delta_n^T)$ has density

$$f_{\Delta_1^T,\dots,\Delta_n^T}(x_1,\dots,x_n)=e^{-\mu(x_1+\ldots+x_n)}\prod_{i=1}^{n}\lambda(x_1+\ldots+x_i),\quad x_i\geq 0.$$

(3) For a mixed Poisson process, the Δ_i^T are mixed exponentials such that each vector $(\Delta_1^T,\dots,\Delta_n^T)$ has density

$$f_{\Delta_1^T,\dots,\Delta_n^T}(x_1,\dots,x_n)=\mathbb{E}\left[\Lambda^n\, e^{-\Lambda(x_1+\ldots+x_n)}\right],\quad x_i\geq 0.$$

(4) For a linear birth process with immigration, the Δ_i^T $(i\geq 1)$ are independent exponentials of parameter $\lambda+\alpha(i-1)$.

(5) For a linear death counting process, the Δ_i^T $(1\leq i\leq z)$ are independent exponentials of parameter $\alpha(z-i+1)$.

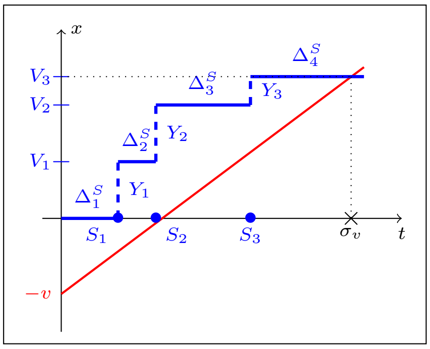
3 Ordered dual risk model

Consider a dual risk model (1) in which the profits arrive according to an OSPP. So, the wealth process is given by

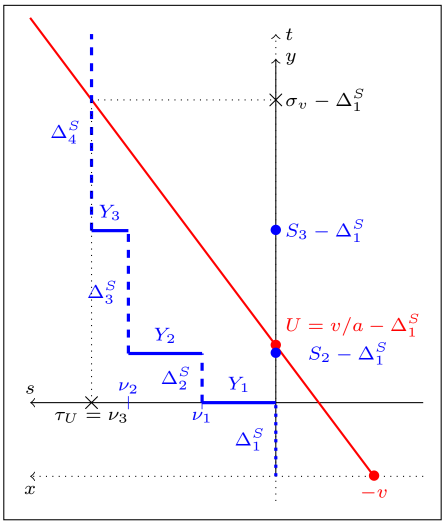
$$W(t)=v-at+V(t),\quad \text{where}\quad V(t)=\sum_{i=1}^{M(t)}Y_i,\quad t\geq 0. \tag{8}$$

Here, $v>0$ and $a>0$, $\{M(t),\, t\geq 0\}$ is an OSPP and the Y_i are i.i.d. non-negative random variables. The partial sums of profits are denoted by $V_n=Y_1+\ldots+Y_n$, $n\geq 1$, with $V_0=0$.

The ruin time σ_v is defined by (2). From (8), we see that $\sigma_v\geq v/a$, i.e. ruin cannot arise before time v/a. Evidently, σ_v is the first-meeting time of the



(a) Ruin time in the dual model. The solid red line represents the cost function $x = -v + at$, and the dashed blue line corresponds to a trajectory of the profit size process $V(t)$.



(b) Ruin time in the insurance model with inverted characteristics. The solid red line represents the premium function $y = U + s/a$ with $U = v/a - \Delta_1^S$, and the dashed blue line corresponds to a trajectory of the claim amount process.

Figure 1: Boundary crossing problem in the dual risk model.

process $\{V(t), t \geq 0\}$ with the lower linear boundary $y = -v + at$, $t \geq 0$. Let S_i, $i \geq 1$, denote the arrival time of the i-th profit, and let $\Delta_i^S = S_i - S_{i-1}$ be the inter-arrival times with $S_0 = 0$. Figure 1(a) illustrates the first-meeting problem under study.

Let us suppose that the Y_i have a density f_Y. As usual, we write f_Y^{*n}, $n \geq 1$, for the n-th convolution of f_Y, with $f_Y^{*0}(y) = \mathbf{1}_{(y=0)}$. Our purpose is to obtain the distribution of the ruin time σ_v. This will be done using a family of Abel-Gontcharoff (A-G) polynomials. A short presentation of the A-G polynomials is given in the Appendix A. They are constructed from a given set of real numbers $U = \{u_i, i \geq 1\}$. The A-G polynomial of degree n in x is then denoted by $G_n(x|U)$.

Theorem 3.1. *The ruin time σ_v takes values $t \geq v/a$. It has an atom at v/a with*

$$\mathbb{P}(\sigma_v = v/a) = \mathbb{P}\left[M(v/a) = 0\right], \tag{9}$$

while for $t > v/a$, it has a density part given by

$$f_{\sigma_v}(t) = a\,\mathbb{E}\left[(-1)^{M(t)}\, f_Y^{*M(t)}(at - v)\, h_{M(t)}(t, v)\mathbf{1}_{\{M(t)\geq 1\}}\right], \tag{10}$$

where if $[M(t) = n]$, the function $h_n(t, v)$ is the conditional expectation

$$h_n(t, v) = \mathbb{E}\left\{G_n\left[0\middle|F_t\left(\frac{V_0 + v}{a}\right), \ldots, F_t\left(\frac{V_{n-1} + v}{a}\right)\right]\middle|V_n = at - v\right\}. \tag{11}$$

Proof. As observed before, ruin is not possible before time v/a. It will occur at time $t = v/a$ if no profits arrived until that time. So, the distribution of σ_v has an atom at $t = v/a$ of probability mass $\mathbb{P}[M(v/a) = 0]$, hence (9).

Now, consider any time $t > v/a$. Evidently, at least one profit is recorded at t (otherwise, ruin would occur at time v/a). Let us look at the event $[\sigma_v \in (t, t + dt)]$ where dt is small enough. We can express it as

$$[\sigma_v \in (t,t+dt)] = \bigcup_{n=1}^{+\infty} \{[M(t) = n] \cap [\sigma_v \in (t,t+dt)]\} \tag{12}$$

$$= \bigcup_{n=1}^{+\infty} \left\{ [M(t)=n] \bigcap_{k=1}^{n} \left[S_k \leq \frac{V_{k-1}+v}{a} \right] \cap \left[\frac{V_n+v}{a} \in (t,t+dt) \right] \right\}. \tag{13}$$

Conditioning on $[M(t) = n]$, (12) gives

$$\mathbb{P}[\sigma_v \in (t,t+dt)] = \sum_{n=1}^{+\infty} \mathbb{P}[\sigma_v \in (t,t+dt) \big| M(t)=n] \mathbb{P}[M(t)=n], \tag{14}$$

and from (13), we get

$$\begin{aligned} &\mathbb{P}[\sigma_v \in (t,t+dt) \big| M(t)=n] \\ &= \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[S_k \leq \frac{V_{k-1}+v}{a} \right] \cap \left[\frac{V_n+v}{a} \in (t,t+dt) \right] \Big| M(t)=n \right\}. \end{aligned} \tag{15}$$

By the order statistic property (see Definition 2.1), given $[M(t) = n]$, the vector $(S_1, \ldots, S_n)$ is distributed as the order statistics $[U_{1;n}(t), \ldots, U_{n;n}(t)]$ of a sample of n i.i.d. random variables with distribution function F_t on $(0, t)$. This implies that

$$[F_t(U_{1:n}(t)), \ldots, F_t(U_{n:n}(t))] \stackrel{\mathcal{D}}{=} (U_{1;n}, \ldots, U_{n;n}), \tag{16}$$

where $(U_{1:n}, \ldots, U_{n:n})$ are the order statistics of n independent uniform variables on $(0, 1)$. Thanks to (16), we may rewrite (15) as

$$\begin{aligned} &\mathbb{P}[\sigma_v \in (t,t+dt) \big| M(t)=n] \\ &= \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[U_{k:n}(t) \leq \frac{V_{k-1}+v}{a} \right] \cap \left[\frac{V_n+v}{a} \in (t,t+dt) \right] \right\} \\ &= \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[U_{k:n} \leq F_t\left(\frac{V_{k-1}+v}{a} \right) \right] \cap \left[\frac{V_n+v}{a} \in (t,t+dt) \right] \right\} \\ &= \mathbb{P}\left\{ \bigcap_{k=1}^{n} \left[U_{k:n} \leq F_t\left(\frac{V_{k-1}+v}{a} \right) \right] \Big| \frac{V_n+v}{a} \in (t,t+dt) \right\} \\ &\qquad \mathbb{P}\left[\frac{V_n+v}{a} \in (t,t+dt) \right]. \end{aligned} \tag{17}$$

The key step here is the probabilistic interpretation of the A-G polynomials given in (A.1). Using that property, we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\bigcap_{k=1}^{n}\left[U_{k:n}\le F_t\left(\frac{V_{k-1}+v}{a}\right)\right]\Big|\frac{V_n+v}{a}\in(t,t+dt)\right\}\\
&=\mathbb{E}\left\{\mathbb{P}\left[\bigcap_{k=1}^{n}\left[U_{k:n}\le F_t\left(\frac{V_{k-1}+v}{a}\right)\right]\Big|V_1,\dots,V_n\right]\Big|V_n=at-v\right\}\\
&=(-1)^n\mathbb{E}\left\{G_n\left[0\Big|F_t\left(\frac{V_0+v}{a}\right),\dots,F_t\left(\frac{V_{n-1}+v}{a}\right)\right]\Big|V_n=at-v\right\}\\
&=(-1)^n h_n(t,v),\quad n\ge 1, \qquad (18)
\end{aligned}$$

in the notation (11). Furthermore, we have

$$\mathbb{P}\left[\frac{V_n+v}{a}\in(t,t+dt)\right]=a\,f_Y^{*n}(at-v)dt,\quad n\ge 1. \tag{19}$$

Inserting (18), (19) in (14), (17), we then deduce the result (10), (11). □

The formulas (10), (11) exhibit clearly the algebraic structure underlying the density part of σ_v. Of course, their numerical implementation can be rather complex but it remains quite practicable. We now show that the result becomes especially simple when the OSPP is a mixed Poisson process (see case (3) of OSPP in Section 2).

Corollary 3.2. *If $\{M(t),\, t\ge 0\}$ is a mixed Poisson process, then*

$$f_{\sigma_v}(t)=\frac{v}{t}\,\mathbb{E}\left[f_Y^{*M(t)}(at-v)\mathbf{1}_{\{M(t)\ge 1\}}\right],\quad t>v/a. \tag{20}$$

Proof. As indicated in Section 2, if the OSPP is a mixed Poisson process, then $F_t(s)=s/t$ for $s\in[0,t]$. Thus, $h_n(t,v)$ in (11) becomes

$$\begin{aligned}
h_n(t,v) &= \mathbb{E}\left[G_n\left(0\Big|\frac{V_0+v}{at},\dots,\frac{V_{n-1}+v}{at}\right)\Big|V_n=at-v\right]\\
&= \frac{1}{(at)^n}\mathbb{E}\left[G_n(-v|V_0,\dots,V_{n-1})\Big|V_n=at-v\right],\quad n\ge 1, \qquad (21)
\end{aligned}$$

using the identity (54) for G_n.

Now, the Y_i being i.i.d. random variables, Property A.2 for the conditional expectation of G_n is applicable to (21). This yields

$$h_n(t,v)=\frac{1}{(at)^n}(-v)(-v-at+v)^{n-1}=(-1)^n\frac{v}{at},\quad n\ge 1. \tag{22}$$

Combining (22) with (10) then gives the formula (20). □

We mention that an analogous result was obtained by Stadje and Zacks [40].

Remark. Consider the special case where $\{M(t),\, t\ge 0\}$ is a standard Poisson process. Then, the process $\{\hat{X}(t)=at-V(t),\, t\ge 0\}$ is a Lévy process which is skip free in the positive direction. In other words, it has no positive jumps and its increments are stationary independent. The stopping time σ_v becomes here $\inf\{t\ge 0 : \hat{X}(t)=v\}$. For this problem, the famous Kendall identity is applicable (see, e.g., Borovkov and Burq [9]). One can check that, as expected, formula (20) is in agreement with that identity.

Examples. (a) Suppose that $\{M(t),\, t \geq 0\}$ is a Poisson process of parameter λ, and that the profit sizes are exponentially distributed of parameter μ. Then, formula (20) becomes explicit as the density part of a compound Poisson distribution with exponentially distributed summands is known (see, e.g., Rolski et al. [35], Equation 4.2.8). This yields

$$f_{\sigma_v}(t) = \frac{v}{t}\, e^{-[\lambda t+\mu(at-v)]} 2\sqrt{\lambda t\mu/(at-v)}\, I_1\left[2\sqrt{\lambda t\mu(at-v)}\right], \quad t > v/a,$$

where $I_1(z)$ is the modified Bessel function $\sum_{k=0}^{+\infty} \frac{(x/2)^{2k+1}}{\Gamma(k+2)}$, $z \in \mathbb{R}$.

(b) Suppose that $\{M(t),\, t \geq 0\}$ is a negative binomial process of parameters $\gamma = 1$ and β (i.e., a mixed Poisson process with an exponential mixing variable of parameter β; see case (3) in Section 2), and that the profit sizes are exponentially distributed of parameter μ. Then, $M(t)$ has a geometric distribution with probability of success $\beta/(t+\beta)$ (see case (3) in Section 2), and formula (20) gives, after some elementary calculations,

$$f_{\sigma_v}(t) = \frac{v\beta\mu}{(t+\beta)^2} \exp\left[-\frac{\beta\mu(at-v)}{t+\beta}\right], \quad t > v/a.$$

The case where γ is any non-negative integer can be handled too as the density part of a compound negative-binomial distribution with exponentially distributed summands is known (e.g., Rolski et al. [35], Equation 4.2.10).

For both examples, the finite-time ruin probability can be easily evaluated through a numerical integration procedure.

4 Primal with claims as spacing levels

Consider an insurance risk model (3) of Sparre-Andersen type in which the claim amounts are distributed as spacing levels in an OSPP. So, the reserve process is given by

$$R(t) = u + ct - C(t), \quad \text{where} \quad C(t) = \sum_{i=1}^{N(t)} X_i, \quad t \geq 0. \tag{23}$$

Here, $u \geq 0$ and $c > 0$, $\{N(t),\, t \geq 0\}$ is a renewal process and the X_i are non-negative random variables with a level spacing distribution. These X_i generate an OSPP $\{M(t),\, t \geq 0\}$, say.

The ruin time τ_u is defined by (4). This time, τ_u is the first-crossing time of the stochastic process $\{C(t),\, t \geq 0\}$ through the upper linear boundary $x = u + ct$, $t \geq 0$. Obviously, τ_u can take any value $t \geq 0$. Let T_i, $i \geq 1$, denote the arrival time of the i-th claim, and let $\Delta_i^T = T_i - T_{i-1}$ be the inter-arrival times with $T_0 = 0$. Figure 2(a) illustrates this first-crossing problem. Note that the crossing is not a meeting as in the previous dual model since the trajectory is jumping over the boundary.

Let us suppose that the Δ_i^T have a density f_{Δ^T}. By a duality argument, we are going to derive a formula for the density of the ruin time τ_u. For that, we will use Theorem 3.1 obtained in Section 3 for the dual risk model. Let us introduce a new variable Δ_0^T distributed as the Δ_i^T and independent of the renewal process.

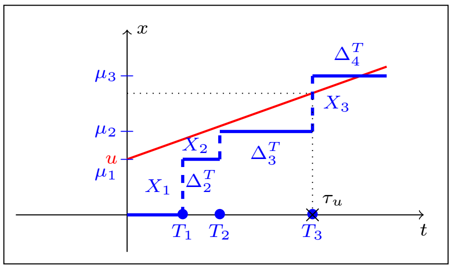
Theorem 4.1. *The ruin time τ_u has a (defective) density at point $t \geq 0$ given by*

$$f_{\tau_u}(t) = \\ \mathbb{E}\left[(-1)^{M(u+ct)} f_{\Delta^T}^{*M(u+ct)}(t-\Delta_0^T)\, h_{M(u+ct)}(u+ct, u/c+\Delta_0^T)\,\mathbf{1}_{\{t\geq\Delta_0^T\}}\right], \tag{24}$$

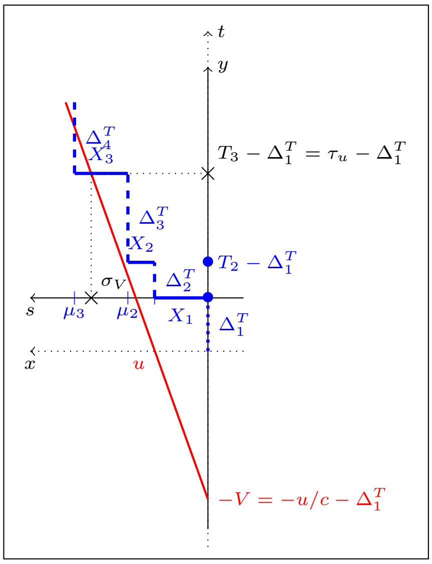
where $h_0(.) = 1$, and if $[M(u+ct) = n]$, $n \geq 1$, and $[\Delta_0^T = d_0]$, $d_0 \geq 0$, the function $h_n(u+ct, u/c+d_0)$ is the conditional expectation

$$h_n\left(u+ct, u/c+d_0\right) = \\ \mathbb{E}\left\{G_n\left[0\middle|F_{u+ct}\left[u+c(T_0+d_0)\right],\ldots,F_{u+ct}\left[u+c(T_{n-1}+d_0)\right]\right]\middle|T_n = t-d_0\right\}. \tag{25}$$

Proof. The first-crossing happens at a jump of the claim amount process, so that deriving the density of τ_u by a direct reasoning is not easy. A simple trick consists in passing to an associated dual model. This approach is rather standard (see the references given in the Introduction). For the sake of clarity, we recall briefly its principle.



(a) Ruin time in the insurance model. The solid red line represents the premium income $x = u + ct$, and the dashed blue line corresponds to a trajectory of the claim amount process $C(t)$.



(b) Ruin time in the dual model with inverted characteristics. The solid red line represents the cost function $y = -V + t/c$ with $V = u/c + \Delta_1^T$, and the dashed blue line corresponds to a trajectory of the profit size process.

Figure 2: Boundary crossing problem in the insurance risk model.

A new origin is put at the point $(0, \Delta_1^T)$ of the original coordinates. Then, an anticlockwise rotation of 90° is made on Figure 2(a). This yields Figure 2(b) in a new system of coordinates, denoted by (s, y). So, we have built a dual risk model whose characteristics are the inverse of those in the insurance model. Its wealth process is given by

$$W(s) = V - \frac{s}{c} + \sum_{i=1}^{M(s)} \Delta_{i+1}^T, \quad s \geq 0, \tag{26}$$

where the cost function is linear with slope $1/c$ and (random) intercept $V = u/c + \Delta_1^T$, the Δ_{i+1}^T become the profits and the X_i correspond to the inter-arrival times in the OSPP $\{M(s), s \geq 0\}$. Note that the random variable V is independent of the Δ_{i+1}^T.

As shown by Figure 2, the first-crossing in the insurance model is equivalent to the first-meeting in the dual model. In fact, we have the simple distributional identity

$$\tau_u - \Delta_1^T =_D \sigma_V/c - V,$$

where σ_V is the first-meeting time in the dual model. From the definition of V, this becomes

$$\tau_u =_D \left(\sigma_{u/c+\Delta_1^T} - u\right)/c. \tag{27}$$

This identity was already obtained in the past; see, e.g., Shi and Landriault [39].

Let us fix $\Delta_1^T = d_1 \,(\geq 0)$. We note that Δ_1^T is independent of the profit process in the dual model (26). So, the conditional distribution of $\sigma_{u/c+d_1}$ is provided by Theorem 3.1. More precisely, we define new random variables $\tilde{T}_i$, $i \geq 1$, as the partial sums $\tilde{T}_i = \sum_{j=1}^i \Delta_{j+1}^T$, with $\tilde{T}_0 = 0$. From (10), (11), we then know that $\sigma_{u/c+d_1} \geq u + cd_1$, with an atom at point $u + cd_1$ of probability mass $\mathbb{P}[M(u + cd_1) = 0]$, and a density part at point $s > u + cd_1$ given by

$$f_{\sigma_{u/c+d_1}}(s|d_1) = \frac{1}{c}\mathbb{E}\left[(-1)^{M(s)} f_{\Delta^T}^{*M(s)}\left(\frac{s}{c} - \frac{u}{c} - d_1\right) h_{M(s)}\left(s, \frac{u}{c} + d_1\right) \mathbf{1}_{\{M(s)\geq 1\}}\right], \tag{28}$$

where if $[M(s) = n]$, $h_n(s, v)$ with $v \equiv u/c + d_1$ is the function

$$h_n(s, v) = \mathbb{E}\left\{G_n\left[0\middle|F_s[c(\tilde{T}_0 + v)], \ldots, F_s[c(\tilde{T}_{n-1} + v)]\right]\middle|\tilde{T}_n = s/c - v\right\}. \tag{29}$$

We now pass to the conditional distribution of τ_u. Given d_1, τ_u takes values $t \geq d_1$. It has an atom at d_1 with probability mass $\mathbb{P}[M(u + cd_1) = 0]$. For $t > d_1$, it has a density part which can be expressed from (27) as

$$f_{\tau_u}(t|d_1) = c\mathbb{E}\left[f_{\sigma_{u/c+d_1}}(u + ct)\right]. \tag{30}$$

Inserting (28) in (30) yields

$$f_{\tau_u}(t|d_1) = \mathbb{E}\left[(-1)^{M(u+ct)} f_{\Delta^T}^{*M(u+ct)}(t - d_1)\, h_{M(u+ct)}\left(u + ct, \frac{u}{c} + d_1\right) \mathbf{1}_{\{M(u+ct)\geq 1\}}\right]. \tag{31}$$

Setting $h_0(.) = 1$, consider the expectation in (31) also in the case where $M(u+ct) = 0$. As $f_{\Delta^T}^{*0}(t - d_1) = \mathbf{1}_{\{t=d_1\}}$, this expectation reduces to $\mathbf{1}_{\{t=d_1\}}\mathbb{P}[M(u +$

$cd_1) = 0]$. Let us collect the previous results for any time $t \geq 0$. We see that they can be gathered into a single formula given by

$$\begin{aligned}
&\mathbb{P}[\tau_u \in (t, t+dt)|d_1]/dt = \\
&\mathbb{E}\left[(-1)^{M(u+ct)} f_{\Delta^T}^{*M(u+ct)}(t-d_1)\, h_{M(u+ct)}\left(u+ct, \frac{u}{c}+d_1\right) \mathbf{1}_{\{t\geq d_1\}}\right].
\end{aligned} \tag{32}$$

The function $h_n(u + ct, u/c + d_1)$, for $n \geq 1$, is provided by (29) in which $s = u + ct$ and $v \equiv u/c + d_1$.

To obtain the distribution of τ_u, it suffices to take the expectation of (32) and (29) with respect to Δ_1^T $(= T_1)$. Note that $\tilde{T}_i = T_{i+1} - T_1$ and $\tilde{T}_i =_D T_i$, $i \geq 1$. Remember also the definition of Δ_0^T (given just before (24)). Using these two points, we then find that the variable τ_u has a (defective) density, $f_{\tau_u}(t)$, given by the formulas (24), (25). □

This result highlights the special algebraic structure underlying the density of τ_u. It is much simplified in the case where the claim amounts are mixed exponentials, i.e when they correspond to the spacing levels in a mixed Poisson process $\{M(t),\, t \geq 0\}$ (see case (3) of level spacings in Section 2).

Corollary 4.2. *If the X_i's are mixed exponentials, then*

$$f_{\tau_u}(t) = \frac{1}{u+ct}\,\mathbb{E}\left[\left(u + c\Delta_0^T\right) f_{\Delta^T}^{*M(u+ct)}(t - \Delta_0^T)\, \mathbf{1}_{\{t \geq \Delta_0^T\}}\right], \quad t \geq 0. \tag{33}$$

Proof. When $M(u+ct) = 0$, (33) is equivalent to (24) (as $h_0(.) = 1$). Consider now $M(u + ct) = n$, $n \geq 1$. By assumption, we know that $F_t(s) = s/t$ for $s \in [0, t]$. Thus, (25) can be rewritten as

$$\begin{aligned}
&h_n\left(u + ct, u/c + d_0\right) \\
&\quad = \mathbb{E}\left\{G_n\left[0\Big|\frac{u + c(T_0 + d_0)}{u+ct}, \ldots, \frac{u + c(T_{n-1} + d_0)}{u+ct}\right]\Big| T_n = t - d_0\right\} \\
&\quad = \left(\frac{c}{u+ct}\right)^n \mathbb{E}\left[G_n\left(-\frac{u}{c} - d_0\Big|T_0, \ldots T_{n-1}\right)\Big| T_n = t - d_0\right], \quad n \geq 1,
\end{aligned} \tag{34}$$

using the relation (54) for G_n.

Since the ΔT_i are i.i.d. variables, Property A.2 can be applied to the conditional expectation of G_n in (34). This gives

$$\begin{aligned}
h_n(u+ct, u/c+d_0) &= [c/(u+ct)]^n \,(-u/c - d_0)\,(-u/c - d_0 - t + d_0)^{n-1} \\
&= (-1)^n (u + cd_0)/(u+ct), \quad n \geq 1.
\end{aligned} \tag{35}$$

Inserting (35) in (24) then yields formula (33). □

From (33), we easily retrieve the result derived by Borovkov and Dickson [10] in the case where the claim sizes are i.i.d. exponentials (see their Theorem 1). In our framework, that means that $\{M(t),\, t \geq 0\}$ is a Poisson process.

Example. Suppose that $\{N(t),\, t \geq 0\}$ is a Poisson process of parameter λ, and that the claim amounts are mixed exponentials with an exponential mixing variable of parameter β. So, the process $\{M(t),\, t \geq 0\}$ is a negative binomial process of parameters $\gamma = 1$ and β (see Example (b) in Section 3). Then, $M(t)$

is geometric with probability of success $\beta/(t+\beta)$, and from (33), the density of τ_u is given explicitly by

$$f_{\tau_u}(t) = \left(\frac{\beta}{u+ct+\beta}\right) f_{\Delta_0^T}(t) + \frac{1}{u+ct}\sum_{n=1}^{\infty}\left(\frac{\beta}{u+ct+\beta}\right)\left(\frac{u+ct}{u+ct+\beta}\right)^n \mathbb{E}\left[(u+c\Delta_0^T) f_{\Delta^T}^{*n}(t-\Delta_0^T)\mathbf{1}_{\{t\geq\Delta_0^T\}}\right]. \tag{36}$$

As Δ_0^T is exponential of parameter λ, the expectation in (36) becomes

$$\begin{aligned}
\mathbb{E}[\ldots] &= \int_0^t (u+cx)\frac{e^{-\lambda(t-x)}(t-x)^{n-1}\lambda^n}{(n-1)!}\lambda e^{-\lambda x}dx \\
&= \frac{\lambda^{n+1}e^{-\lambda t}}{(n-1)!}\int_0^t (u+cx)(t-x)^{n-1}dx \\
&= \frac{\lambda^{n+1}e^{-\lambda t}}{(n-1)!}\left(\frac{ut^n}{n}+\frac{ct^{n+1}}{n(n+1)}\right).
\end{aligned} \tag{37}$$

Inserting (37) in (36) then yields

$$\begin{aligned}
f_{\tau_u}(t) &= \frac{\beta\lambda}{u+ct+\beta}e^{-\lambda t} + \frac{\beta\lambda u}{(u+ct)(u+ct+\beta)}\left(e^{-\lambda t\beta/(u+ct+\beta)} - e^{-\lambda t}\right) \\
&+ \frac{\beta c}{(u+ct)^2}\left(e^{-\lambda t\beta/(u+ct+\beta)} - e^{-\lambda t} - \lambda t\frac{u+ct}{u+ct+\beta}e^{-\lambda t}\right).
\end{aligned}$$

5 Dual with profits as spacing levels

Let us go back to a dual risk model (1) of Sparre-Andersen type in which the profit sizes are distributed as spacing levels in an OSPP. The wealth process has still the form (8):

$$W(t) = v - at + V(t), \quad \text{where} \quad V(t) = \sum_{i=1}^{M(t)} Y_i, \quad t \geq 0, \tag{38}$$

but this time, $\{M(t),\, t \geq 0\}$ is a renewal process and the Y_i are non-negative random variables, with a level spacing distribution. These Y_i's generate an OSPP $\{N(t),\, t \geq 0\}$, say. As in Section 3, the profit arrival times are denoted by S_i, and the inter-arrival times by Δ_i^S, $i \geq 1$, with $S_0 = 0$.

In insurance, the ruin probability represents an important risk measure. Over an infinite horizon, it has been the object of many research works. For practical purposes, it is more appropriate to consider a finite horizon. Recently, Lefèvre and Picard [28] determined the finite-time ruin probability for the ordered insurance model with i.i.d. claim amounts. Our purpose here is to obtain, by duality, the finite-time ruin probability in the dual model above.

Let us define $\varphi_v(t)$ as the probability of non-ruin until time t, i.e.,

$$\varphi_v(t) = \mathbb{P}(\sigma_v > t), \quad t \geq 0. \tag{39}$$

Obviously, $\varphi_v(t) = 1$ if $t < v/a$. Thus, from now on, we suppose that $t \geq v/a$. In the next proposition, we will express $\varphi_v(t)$ in terms of a family of Appell polynomials. A short presentation of these polynomials is given in the Appendix A. The Appell and A-G polynomial families are closely related but different. Given a set of real numbers $U = \{u_i, \, i \geq 1\}$, the Appell polynomial of degree n in x is denoted by $A_n(x|U)$. As for Theorem 4.1, we introduce a new variable Δ_0^S distributed as the Δ_i^S and independent of the renewal process.

Theorem 5.1. *For $t \geq v/a$, the non-ruin probability $\varphi_v(t)$ can be expressed as*

$$\varphi_v(t) = \mathbb{E}\left[g_{N(at-v)}(at-v, v/a - \Delta_0^S)\, \mathbf{1}_{\{\Delta_0^S \leq v/a,\, S_{N(at-v)} \leq t - \Delta_0^S\}}\right], \tag{40}$$

where if $[N(at-v) = n]$, $n \geq 0$, and $[\Delta_0^S = d_0]$, $0 \leq d_0 \leq v/a$, the function $g_n(at-v, v/a - d_0)$ is the conditional expectation

$$\begin{aligned} & g_n(at-v, v/a - d_0) \\ & = \mathbb{E}\left\{A_n\left[1 \middle| F_{at-v}\left[a(S_1 + d_0 - v/a)_+\right], \ldots, F_{at-v}\left[a(S_n + d_0 - v/a)_+\right]\right]\right\}. \end{aligned} \tag{41}$$

Proof. We apply duality to convert the first-meeting problem for the dual model (38) into an equivalent first-crossing problem for the insurance model with inverted characteristics. Such a reasoning was followed e.g. by Dimitrova et al. [18]. The primal result used here is a simple and compact formula that was obtained by Lefèvre and Picard [28] for the ordered insurance model.

Let us consider Figure 1(a). A new system of coordinates, denoted by (s, y), is constructed with its origin put at the point $(0, \Delta_1^S)$ of the original coordinates. Then, an anticlockwise rotation of 90° is made, which yields Figure 1(b). In that way, we have constructed an insurance model whose characteristics are the inverse of those in the dual model. Its reserve process is defined as

$$R(s) = U + \frac{s}{a} - \sum_{i=1}^{N(s)} \Delta_{i+1}^S, \tag{42}$$

where the premium rate is equal to $1/a$, the initial reserves are of (random) amount $U = v/a - \Delta_1^S$, the Δ_{i+1}^S become the claim amounts and the profits Y_i correspond to the inter-arrival times in the OSPP $\{N(s), \, s \geq 0\}$. Observe that the random variable U is independent of the Δ_{i+1}^S.

From Figure 1, the first-meeting in the dual model is equivalent to the first-crossing in the insurance model. This gives the distributional identity

$$\sigma_v - \Delta_1^S =_D \tau_U/a + U,$$

where τ_U is the first-crossing time in the insurance model. Using the definition of U, we have

$$\sigma_v =_D (\tau_{v/a - \Delta_1^S} + v)/a. \tag{43}$$

Let $\phi_U(s) = \mathbb{P}(\tau_U > s)$, the probability of non-ruin until time s in the insurance model. From (39) and (43), we can rewrite the probability $\varphi_v(t)$ as

$$\begin{aligned} \varphi_v(t) &= \mathbb{P}\left(\tau_{v/a - \Delta_1^S} > at - v\right) \\ &= \mathbb{E}\left[\phi_{v/a - \Delta_1^S}(at - v)\right], \end{aligned} \tag{44}$$

where the expectation is taken with respect to the variable Δ_1^S $(= S_1)$. Note that if $\Delta_1^S > v/a$, the initial reserves are negative so that ruin arises at time 0.

For the insurance model (42) where $\{N(s),\, s \geq 0\}$ is an OSPP, Lefèvre and Picard [28] derived a close formula for the probability of non-ruin over a finite horizon. In the present case, however, we have to take into account that the initial reserves $U = v/a - \Delta_1^S$ are random and can be negative. So, let $\tilde{S}_i$ denote the partial sums $\tilde{S}_i = \sum_{j=1}^{i} \Delta_{j+1}^S$, $i \geq 1$, with $\tilde{S}_0 = 0$. Making a simple adaptation of that formula, we can then express the non-ruin probability until time s as

$$\mathbb{E}[\phi_U(s)] = \mathbb{E}\left[g_{N(t)}(s,U)\, \mathbf{1}_{\{U\geq 0, \tilde{S}_{N(s)} \leq U+s/a\}}\right], \tag{45}$$

where if $[N(s) = n]$, $n \geq 0$, and $[U = u]$, $u \geq 0$, $g_n(s,u)$ is given by

$$g_n(s,u) = \mathbb{E}\left\{A_n\left[1\middle|F_s[a(\tilde{S}_1 - u)_+], \ldots, F_s[a(\tilde{S}_n - u)_+]\right]\right\}. \tag{46}$$

From (44), (45) and (46), we deduce that

$$\varphi_v(t) = \mathbb{E}\left[g_{N(at-v)}(at - v, v/a - \Delta_1^S)\, \mathbf{1}_{\{\Delta_1^S \leq v/a,\, \tilde{S}_{N(at-v)} \leq t - \Delta_1^S\}}\right], \tag{47}$$

where if $[N(at - v) = n]$, $n \geq 0$, and $[\Delta_1^S = s_1]$, $0 \leq s_1 \leq v/a$,

$$\begin{aligned} &g_n(at - v, v/a - s_1) = \\ &\quad \mathbb{E}\left\{A_n\left[1\middle|F_{at-v}[a(\tilde{S}_1 - v/a + s_1)_+], \ldots, F_{at-v}[a(\tilde{S}_n - v/a + s_1)_+]\right]\right\}. \end{aligned} \tag{48}$$

Finally, as $\tilde{S}_i = S_{i+1} - S_1 =_D S_i$, $i \geq 1$, and using the definition of Δ_0^S, we see that (47), (48) match formulas (40), (41). □

Theorem 5.1 compares with Proposition 2.2 obtained by Dimitrova et al. [18] for another dual risk model in which the profit arrival process is arbitrary and the profit sizes are linear combinations of independent exponentials.

6 Concluding remarks

Our study deals with three ruin problems for different risk models with dependence. The first risk process is a dual model where the profit sizes are i.i.d. but their arrival process satisfies the order statistic property. Using a direct analysis, we obtain the ruin time distribution for the model. The second risk process is a Sparre-Andersen insurance model where claims arrive according to a renewal process and their amounts have a level spacing distribution. We derive here the (defective) ruin time density by applying a duality argument to the previous ruin problem. The third risk process is a dual model but where the profit sizes have a level spacing distribution and their arrival is described by a renewal process. Using again a duality argument, we obtain the probability of non-ruin over a finite horizon. In all cases, the formulas have a clear algebraic structure and can be used for numerical computation. Some illustrations for simpler variants of these models can be found, e.g., in Goffard and Lefèvre [23] and Dimitrova et al. [18]. Of course, other ruin topics could be studied by a similar duality approach. To close, we mention that an alternative approach to such ruin problems consists in working with Laplace transforms. This is the method followed e.g. in Perry et al. [32] and the references therein.

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A Appell and A-G polynomials

Appell and Abel-Gontcharov (A-G) polynomials are well-known in mathematics. They can be used to solve various problems in statistics and applied probabilty. A short presentation is provided below. We refer e.g. to Lefèvre and Picard [30] for a review with applications in risk modelling.

Let $U = \{u_i,\, i \geq 1\}$ be a sequence of reals, non-decreasing in our context. To U is attached a (unique) family of Appell polynomials of degree n in x, $\{A_n(x|U),\, n \geq 0\}$, defined as follows. Starting with $A_0(x|U) = 1$, the $A_n(x|U)$ satisfy the differential equations

$$A_n^{(1)}(x|U) = nA_{n-1}(x|U),$$

with the border conditions

$$A_n(u_n|U) = 0, \quad n \geq 1. \tag{49}$$

So, each A_n, $n \geq 1$, has the integral representation

$$A_n(x|U) = n! \int_{u_n}^{x} \left[\int_{u_{n-1}}^{y_n} \mathrm{d}y_{n-1} \ldots \int_{u_1}^{y_1} \mathrm{d}y_1 \right] \mathrm{d}y_n. \tag{50}$$

In parallel, to U is attached a (unique) family of Abel-Gontcharov (A-G) polynomials of degree n in x, $\{G_n(x|U),\, n \geq 0\}$. Starting with $G_0(x|U) = 1$, the $G_n(x|U)$ satisfy the differential equations

$$G_n^{(1)}(x|U) = nG_{n-1}(x|\mathcal{E}U),$$

where $\mathcal{E}U$ is the shifted family $\{u_{i+1},\, i \geq 1\}$, and with the border conditions

$$G_n(u_1|U) = 0, \quad n \geq 1.$$

So, each G_n, $n \geq 1$, has the integral representation

$$G_n(x|U) = n! \int_{u_1}^{x} \left[\int_{u_2}^{y_1} \mathrm{d}y_2 \ldots \int_{u_n}^{y_{n-1}} \mathrm{d}y_n \right] \mathrm{d}y_1. \tag{51}$$

Note that both polynomial families are sometimes defined without the factor $n!$ in (50) and (51). Of course, these polynomials are related through the identity

$$G_n(x|u_1, \ldots, u_n) = A_n(x|u_n, \ldots, u_1), \quad n \geq 1. \tag{52}$$

However, the two families (i.e. considered for all $n \geq 0$) are distinct and enjoy quite different properties.

Now, from (50) and (51), we directly see that the polynomials A_n and G_n, $n \geq 1$, can be interpreted in terms of the joint distribution of the order statistics $(U_{1:n}, \ldots, U_{n:n})$ of a sample of n independent uniform random variables on $[0, 1]$.

Proposition A.1. *For* $0 \le u_1 \le \ldots \le u_n \le x \le 1$*,*

$$P[U_{1:n} \ge u_1, \ldots, U_{n:n} \ge u_n \;\; \text{and} \;\; U_{n:n} \le x] = A_n(x|u_1, \ldots, u_n),$$

while for $0 \le x \le u_1 \le \ldots \le u_n \le 1$*,*

$$P[U_{1:n} \le u_1, \ldots, U_{n:n} \le u_n \;\; \text{and} \;\; U_{1:n} \ge x] = (-1)^n \, G_n(x|u_1, \ldots, u_n). \quad (53)$$

These representations play a key role in the first-passage problems discussed in the paper. We will also use the simple relation

$$A_n(x|a + bU) = b^n A_n \left[(x - a)/b \, |U\right], \quad n \ge 1, \quad (54)$$

and similarly for G_n. An important particular case in our study is when the parameters in U are random and correspond to partial sums of exchangeable random variables.

Proposition A.2. *Let* $\{X_n \; ; n \ge 1\}$ *be a sequence of exchangeable random variables, of partial sums* $S_n = \sum_{k=1}^n X_k$ *with* $S_0 = 0$*. Then, for* $n \ge 1$*,*

$$\mathbb{E}\left[A_n(x|S_1, \ldots, S_n)|S_n\right] = x^{n-1}(x - S_n), \quad (55)$$

$$\mathbb{E}\left[G_n(x|S_0, \ldots, S_{n-1})|S_n\right] = x(x - S_n)^{n-1}. \quad (56)$$

Proof. The identity (55) was derived in Proposition A.1 of Lefèvre and Picard [28]. For (56), we write

$$\begin{aligned}\mathbb{E}\left[G_n(x|S_0, \ldots, S_{n-1})|S_n\right] &= \mathbb{E}\left[G_n(x - S_n|S_0 - S_n, \ldots, S_{n-1} - S_n)|S_n\right] \\ &= (-1)^n \mathbb{E}\left[G_n(S_n - x|S_n, \ldots, S_n - S_{n-1})|S_n\right] \\ &= (-1)^n \mathbb{E}\left[A_n(S_n - x|S_n - S_{n-1}, \ldots, S_n)|S_n\right], \quad (57)\end{aligned}$$

using successively the relations (54) and (52). As the X_n's are exchangeable, we deduce from (57) and (55) the desired formula (56). □

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