# Polynomial approximations suited for compound distributions: Applications to insurance

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## 1 Introduction

A random variable $X = \sum_{i=1}^{N} U_i$, where $N$ is a counting random variable and $(U_i)_{i\in\mathbb{N}}$ is a sequence of **i.i.d.** non-negative continuous random variables independent of $N$, is governed by a compound distribution. Actuaries often study these distributions as they play a key role in non-life insurance risk management. For instance, the random variable $X$ may model the total claim amounts associated to a non-life insurance portfolio over a given time period. The infinite time horizon ruin probability in the compound Poisson ruin model is the survival function of a random variable $X$, where $N$ is geometrically distributed. The probability density function of $X$ admits a tractable expression in only a few cases, which makes it difficult to compute probabilities of interest for risk management purposes. Although it is possible to envisage the use of Monte Carlo simulations, the computing time to reach an acceptable accuracy is too considerable to make it a viable solution. This remark justifies the use of numerical methods to approximate probabilities. A multidimensional extension, where a random vector $\mathbf{X}$ is considered, allows to jointly model all risks and permits to capture the dependence among them. The increase of the dimension automatically enhances the complexity of the problem and the need for numerical alternatives. The polynomial approximation at the heart of this work belongs to these numerical methods.

A great body of the actuarial science litterature focuses on the derivation of approximations techniques. They usually fall into two groups. The methods that imply the arithmetization of the claim sizes distribution and those for which this preliminary step is not needed. One of the oldest method is based on a recursive algorithm due to Panjer [29]. The historical competitor to Panjer's algorithm is the so called Fast Fourier Transform (FFT) algorithm which performs a numerical inversion of the discrete Fourier transform, see for instance [16]. The performance of these methods have been compared in several research works such as [9], their main drawback lies in the mandatory discretization of the claim amounts distribution. As the polynomial approximation does not include this step, it seems fairer to offer a comparison to more recent techniques based on the numerical inversion of the Laplace transform. Comparisons to the direct Fourier transform inversion described in the Chapter $5$ of [1] and to the inversion of the scaled Laplace transform [26] are proposed in the section devoted to numerical illustrations.

This paper discusses a method to approximate probability density function $f_X$ via polynomials $(Q_n)_{n\in\mathbb{N}}$ which are orthonormal with respect to a reference probability measure $\nu$. The approximation takes the form

$$\widehat{f}_X(x) = \sum_{k=0}^{K} a_k Q_k(x) f_\nu(x), \tag{1}$$

where the expansion's coefficients $(a_k)_{k\in\mathbb{N}}$ are linear combinations of moments of $X$, $K$ is the order of truncation, and $f_\nu$ denotes the **p.d.f.** of $\nu$. The method extends naturally within a multivariate framework to recover joint probability density functions associated to random vectors. The choice of the reference distribution, the validity of the polynomial expansion and the accuracy given $K$ are discussed.

In Section 2 different actuarial science problems where compound distributions arise are exposed, Section 3 describes the polynomial approximation and discusses the application to compound distribution, Section 4 provides a brief presentation of the competitors of the polynomial approximation, and Section 5 is devoted to numerical illustrations, the polynomial approximation is employed to cope with the problems listed in Section 2.

## 2 Actuarial science problems linked to compound distributions

### 2.1 Collective model in dimension $1$

Consider a non-life insurance portfolio, like third-party liability motor insurance for instance, over a given time period, say a year. The number of claims is unknown and modeled through a counting random variable $N$ governed by a discrete probability measure $\mathbb{P}_N$ with associated **p.d.f.**

$$f_N(k) = \mathbb{P}(N = k), \text{ for } k = 1, 2, \ldots.$$

Each claim causes a loss for the insurance company. These losses are assumed to be independent and therefore modeled by a sequence of **i.i.d.** non-negative random variables governed by a continuous probability measure $\mathbb{P}_U$ having **p.d.f.** $f_U$. The underlying risks for the whole portfolio is quantified by

$$X = \sum_{i=1}^{N} U_i, \tag{2}$$

which is a random variable governed by a compound distribution $(\mathbb{P}_N; \mathbb{P}_U)$. The probability measure associated to a compound distribution is a mixed probability measure

$$\mathrm{d}\mathbb{P}_X(x) = f_N(0)\delta_0(x) + \mathrm{d}\mathbb{G}_X(x), \tag{3}$$

with a singular part $f_N(0)\delta_0(x)$ due to the non-null probability mass at $0$, note that $\delta_0$ is the Dirac measure at $0$ and an absolutely continuous part[1] $\mathbb{G}_X$ which is a defective probability measure[2]. The **p.d.f.** of $\mathbb{G}_X$ is given by

$$g_X(x) = \sum_{k=1}^{+\infty} f_N(k) f_U^{*k}(x), \tag{4}$$

after conditioning with respect to $N$. The expression of the **p.d.f.** (4) is problematic because it involves an infinite series whose summands are repeated convolution of $f_U$ with itself[3] which can be painful to evaluate. The **p.d.f.** of $X$ is available in a closed form in only a few cases[4]. The model specified by (2) is called the collective model and can be found in many texbooks [5, 30, 21]. The knowledge of the distribution of $X$ enables actuaries to compute interesting quantities like quantiles or premiums. The generalized stop-loss premium is defined as

$$\Pi_{c,d}(X) = \mathbb{E}\left[(X - c)_+^d\right], \tag{5}$$

[1] Absolutely continuous with respect to the Lebesgue measure.
[2] A defective probability measure integrates to $p \leq 1$ instead of $1$.
[3] Note that $f_U^{*k}$ is the **p.d.f.** of the random sum $\sum_{i=1}^{k} U_i$.
[4] One example is when $N$ is Poisson distributed and the $U_i$'s are exponentially distributed.

where the function $x \mapsto (x)_+$ is the positive part, $c \geq 0$, and $d \in \mathbb{N}$. The generalized stop-loss premium is related to well known premiums in actuarial science,

- The pure premium is obtained when setting $c = 0$ and $d = 1$,

$$\Pi_{0,1}(X) = \mathbb{E}(X).$$

- The usual stop-loss premium[5] is obtained when setting $d = 1$,

$$\Pi_{1,c}(X) = \mathbb{E}\left[(X - c)_+\right].$$

In this particular application the survival function $\overline{F}_X(x) = \mathbb{P}(X > x)$ is even more relevant than the **p.d.f.**. Namely, every generalized stop-loss premium (5) is computable through the following integral formula

$$\Pi_{c,d} = \int_0^{+\infty} d \times y^{d-1} \overline{F}_X(c + y) \mathrm{d}y.$$

An insurance company is critically intent on having a positive financial reserve over time (which is also important from the regulators' point of view). The ruin probability can be defined in this context as the probability of a negative technical result, that corresponds to a situation where the aggregated claim amounts exceeds the accumulated premium

$$\psi(u) = \mathbb{P}(X - P > u), \tag{6}$$

where $P$ represents the accumulated premium collected by the insurance company. The quantity $u$ is interpretated as a solvency capital required to achieve a predefined risk level target $\alpha$. In fact, $u$ is the solution of the equation $\psi(u) = \alpha$ which relates it to the so-called Value-at-Risk $u_\alpha = \mathrm{VaR}_X(1 - \alpha) - P$. At this stage one may note that the knowledge of the distribution of $S$ is useful for both computing premiums and doing claim reserving. The next subsection describes a multi-dimensional extension of the collective model. Subsection 5.1 offers an illustration where the **p.d.f.**, the survival function, the usual stop-loss premium, and a more practical stop-loss premium are calculated in the case of a compound Poisson distribution[6].

## 2.2 Collective model in dimension $n$

In insurance and in reinsurance, some common events may cause simultaneous, correlated claims in two lines of business. In third-party liability motor insurance, an accident may cause corporal damage as well as material damage losses. These parts of the same claim are then handled by different claim managers and reserving is most often also done separately, which makes it necessary to study the joint distribution of the vector $\mathbf{X} = (X_1, \ldots, X_n)$ whose components are the total claim amounts of the portfolios asociated to the different lines of business. Similarly, a reinsurer who accepts two stop-loss treaties from two customers operating on the same market is exposed

[5] The usual stop-loss premium is associated to a global, non-proportional reinsurance treaty.

[6] A compound Poisson distribution corresponds to the particular case where $N$ is Poisson distributed.

to the sum of two excesses of aggregate losses, usually divided into an independant part and a common shock part arising from events that generates claims for both of the insurance companies at the same time. To compute the **VaR** of the reinsurer's exposure we need the bivariate distribution of the aggregate claim amounts of the two insurers.

The multivariate collective model is characterized by the vector

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{N_1} U_{1j} \\ \vdots \\ \sum_{j=1}^{N_n} U_{nj} \end{pmatrix} + \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ \vdots \\ V_{nj} \end{pmatrix}, \tag{7}$$

whose components represent the total claim amounts associated to each insurance portfolio. The vector $\mathbf{X} = (X_1, \ldots, X_n)$ is governed by a multivariate compound distribution

$$\left( [\mathbb{P}_{\mathbf{N}}, \mathbb{P}_M] \left[ \mathbb{P}_{U_1}, \ldots, \mathbb{P}_{U_n}, \mathbb{P}_{\mathbf{V}} \right] \right),$$

where $\mathbf{N} = (N_1, \ldots, N_n)$ is a counting random vector governed by a multivariate discrete distribution $\mathbb{P}_{\mathbf{N}}$ allowing for dependencies among the frequencies, $(U_{ij})_{j \in \mathbb{N}}$ are sequences of continuous, non-negative, **i.i.d.** and $\mathbb{P}_{U_i}$ distributed random variables for $i = 1, \ldots, n$, $M$ is a counting random variable having a discrete distribution $\mathbb{P}_M$, and $(\mathbf{V}_j)_{j \in \mathbb{N}}$ is a sequence of **i.i.d.** random vectors having a multivariate continuous distribution $\mathbb{P}_{\mathbf{V}}$ that permits to include dependency among the claim severities. The model (7) extends the model presented in a recent research paper [19] where the two dimensional case is considered. Deriving closed formulas for the joint **p.d.f.** of multivariate compound distribution is even more difficult than in one dimension, which makes it desirable to develop efficient numerical approximation techniques. The methods mentioned in Subsection 2.1 admits extension to $n$ dimensions. Recursive methods of Panjer's algorithm type are at the center of many research works [17, 31, 35, 32]. Comparisons to FFT algorithm in multivariate framework can be found in [18]. The main drawback of these approaches remains the arithmetization of the claim amounts distribution. The inversion of the multidimensional Laplace transform is quite a new topic, and it is rare in the literature. The direct multivariate Fourier transform inversion is described in [6], and the exponential moments method is extended in [25]. Note that neither of these works recovers the joint **p.d.f.** of a random variable defined as in (7).

The next example aims at demonstrating the relevance of approximating the joint **p.d.f.** of (7) for reinsurance risk management purposes.

**Example 2.1.** A reinsurance company proposes a global, non-proportional, reinsurance treaty to several insurance companies. The portfolios under reinsurance are associated to the same line of business[7] and to total claim amounts stored in the vector $\mathbf{X} = (X_1, \ldots, X_n)$. The reinsurer offers non-proportional reinsurance treaties characterized by their priorities $\mathbf{c} = (c_1, \ldots, c_n)$ and limits $\mathbf{d} = (d_1, \ldots, d_n)$. At the end of the day, the risk exposure of the reinsurer is given by

$$Z = \sum_{i=1}^{n} \max\left[X_i - c_i\,,\, d_i\right]. \tag{8}$$

[7] but belong to different insurance companies

It is desirable to have access to the quantiles of the distribution of $Z$ to determine solvability margins over a given time period which necessitates the knowledge of the distribution of the vector $\mathbf{X}$.

Subsection 5.2 illustrates the performance of the orthogonal polynomials method to recover the joint survival function of the random vector (7)[8], and show how to use it in order to study the risk profile (8) of a reinsurer in presence of two non-proportional treaties. Those results are extracted from a published research work [13].

## 2.3 Ultimate ruin probability in the compound Poisson ruin model

Ruin theory aims at adding a dynamic flavour in risk modeling, via the introduction of continuous stochastic processes to model the financial reserves of an insurance company over time. A non-life insurance company is assumed to be able to follow the financials' evolution associated with one of its portfolios in continuous time. The number of claims until time $t$ is modeled through a homogeneous Poisson process $\{N_t\}_{t\geq 0}$ with intensity $\lambda$. The successive claim amounts $(U_i)_{i\in\mathbb{N}}$, form a sequence of positive, **i.i.d.**, continuous random variables, and independent from $\{N_t\}_{t\geq 0}$. The initial reserves are $u \geq 0$, and the premium rate is constant and equal to $p \geq 0$. The risk reserve process is defined as

$$R_t = u + pt - \sum_{i=1}^{N_t} U_i. \tag{9}$$

This model is called the compound Poisson ruin model[9]. The associated claims surplus process is defined as $S_t = u - R_t$. One quantity of interest is the ultimate ruin probabilities defined as

$$\psi(u) = \mathbb{P}\left(\inf_{t\geq 0} R_t < 0 \middle| R_0 = u\right) = \mathbb{P}\left(\sup_{t\geq 0} S_t > u \middle| S_0 = 0\right). \tag{10}$$

It is worth noting the simplicity of model (9) and insisting on the interpretation of the ruin probability as a risk measure to assess the long term viability of the premium and the amounts of the initial reserves. Namely, the initial reserves are set to keep the risk (**e.g.** ruin probability) below some prescribed threshold. The premium rate $p$ is given by

$$p = (1+\eta)\lambda\mathbb{E}(U), \tag{11}$$

where $\eta > 0$ is the safety loading[10] that indicates how much the collected premium exceeds the average claim amounts per unit of time. An important result regarding the ultimate ruin probability within the frame of the compound Poisson ruin model is the Pollaczek-Khinchine formula that expresses the ultimate ruin probability as the survival function of a geometric compound distribution

$$\psi(u) = \mathbb{P}\left(\sum_{i=1}^{N} V_i > u\right), \tag{12}$$

[8] A simplified two-dimensional version of it.

[9] Also known as Cramér-Lundberg ruin model or classical ruin model.

[10] Usually expressed in percentages.

where $N$ is geometrically distributed $\mathcal{NB}\left(1, \frac{\lambda \mathbb{E}(U)}{p}\right)$[11], and the $V_i$'s form a sequence of continuous, non-negative, **i.i.d.** random variables having a **p.d.f.** $f_V(x) = \frac{\overline{F_U}(x)}{\mathbb{E}(U)}$. The problem of approximating the ultimate ruin probability reduces to approximating the survival function of a special compound distribution for which the techniques outlined in Subsection 2.1 are well suited for the job. Panjer's algorithm has been applied to the problem in [7], the exponential moments methods is employed in [27] and the direct Fourier transform inversion is performed in the chapter 5 of [30]. Of course, there are many other research papers that deal with ruin probability approximation, we simply mentioned those describing the competitors of the orthogonal polynomial approximation implemented in the section dedicated to numerical illustrations. A comprehensive view of ruin theory is available in the books [15, 2, 30].

Subsection 5.3 shows the performance of the polynomial approximation of the ultimate ruin probability in the compound Poisson ruin model, complementary results are available in [14].

# 3 Orthonormal polynomial approximations for probability density function

## 3.1 General formulas

Let $X$ be a random variable which has a density $f_X$ with respect to some measure $\lambda$[12]. If $f_X$ is unknown but the distribution of $X$ is expected to be close to some probability measure $\nu$ with **p.d.f.** $f_\nu$, one may use $f_\nu$ as a first approximation to $f_X$ and try to improve it by invoking suitable correction terms. Assuming that all moments of $\nu$ are finite, the standard Gram-Schmidt orthogonalization technique shows the existence of a set of polynomials $\{Q_k\}_{k\in\mathbb{N}}$[13] which are orthonormal in $\mathscr{L}^2(\nu)$[14] equipped with the usual inner product $\langle g, h\rangle = \int g(x)h(x)\mathrm{d}\nu(x)$ and the corresponding norm $||g||^2 = \langle g, g\rangle$. That is, the $Q_k$'s satisfy

$$\langle Q_k, Q_l\rangle = \int Q_k(x)Q_l(x)\mathrm{d}\nu(x) = \delta_{kl},\ k, l \in \mathbb{N}, \tag{13}$$

where $\delta_{kl}$ denotes the Kronecker symbol equal to 1 when $k = l$ and 0 otherwise. If there exists an $\alpha > 0$ such that

$$\int e^{\alpha|x|}\mathrm{d}\nu(x) < \infty, \tag{14}$$

then the set $(Q_k)_{k\in\mathbb{N}}$ is complete in $\mathscr{L}^2(\nu)$, see Chapter 7 of the book by Nagy [28]. If the **p.d.f.** of $X$ with respect to $\nu$[15], denoted by $\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\nu}$, belongs to $\mathscr{L}^2(\nu)$, that is if

$$\int \left(\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\nu}(x)\right)^2 \mathrm{d}\nu(x) < +\infty, \tag{15}$$

[11] $\mathcal{NB}(a, \rho)$ denotes the negative binomial distribution with parameters $a$ and $\rho$.
[12] Typically Lebesgue measure on an interval or counting measure on a subset of $\mathbb{Z}$.
[13] $k$ indicates the degree of the polynomial.
[14] $\mathscr{L}^2(\nu)$ denotes the set of square integrable functions with respect to $\nu$.
[15] The absolute continuity of $\mathbb{P}_X$ with respect to $\nu$ is taken for granted.

then $\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\nu}$ is expanded, through orthogonal projection onto $\{Q_k\}_{k\in\mathbb{N}}$, as

$$\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\nu}(x) = \sum_{k=0}^{+\infty} a_k Q_k(x), \tag{16}$$

where the coefficients of the expansion are given by

$$a_k = \left\langle \frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\nu}, Q_k \right\rangle = \int Q_k(x) \frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\nu}(x) \mathrm{d}\nu(x) = \mathbb{E}\left[Q_k(X)\right], \text{ for } k \in \mathbb{N} \tag{17}$$

The final approximation of the **p.d.f.** $f_X$ follows from truncation

$$f_X^K(x) = \sum_{k=0}^{K} a_k(x) Q_k(x) f_\nu(x), \tag{18}$$

where $K$ denotes the order of truncation. The accuracy of the polynomial approximation (18) relies on the behavior of the sequence $(a_k)_{k\in\mathbb{N}}$, and in particular the speed of their decay. Note that the Parseval relationship

$$\int \left(\frac{\mathrm{d}\mathbb{P}_X}{\mathrm{d}\nu}(x)\right)^2 \mathrm{d}\nu(x) = \sum_{k=0}^{+\infty} a_k < \infty, \tag{19}$$

holds and implies that $\lim_{k\to+\infty} a_k = 0$.

The method extends naturally to higher dimensions. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random vector governed by a multivariate probability measure $\mathbb{P}_\mathbf{X}$ having a **p.d.f.** with respect to a multivariate positive measure $\boldsymbol{\lambda}$. Consider a multivariate probability measure

$$\boldsymbol{\nu}(\mathbf{x}) = \nu(x_1, \ldots, x_n) = \prod_{i=1}^{n} \nu_i(x), \tag{20}$$

built from the product of univariate probability measures $\nu_i$. Let $(Q_k^{\nu_i})$ be the sequence of orthonormal polynomials with respect to $\nu_i$ for $i = 1, \ldots, n$. The multivariate polynomials

$$Q_\mathbf{k}(\mathbf{x}) = \prod_{i=1}^{n} Q_{k_i}^{\nu_i}(x_i), \tag{21}$$

where $\mathbf{k} = (k_1 \ldots, k_n)$, are orthonormal with respect to $\boldsymbol{\nu}$. Futhermore, the sequence $\{Q_\mathbf{k}\}_{\mathbf{k}\in\mathbb{N}^n}$ is complete in $\mathscr{L}^2(\boldsymbol{\nu})$ if there exists $\boldsymbol{\alpha} \in \mathbb{R}_+^n$ such that

$$\int e^{\langle \boldsymbol{\alpha}, \mathbf{x} \rangle} \mathrm{d}\boldsymbol{\nu}(\mathbf{x}) < \infty, \tag{22}$$

where $\langle \boldsymbol{\alpha}, \mathbf{x} \rangle = \sum_{i=1}^n \alpha_i x_i$ is the usual inner product in $\mathbb{R}^n$. If $\frac{\mathrm{d}\mathbb{P}_\mathbf{X}}{\mathrm{d}\boldsymbol{\nu}} \in \mathscr{L}^2(\boldsymbol{\nu})$ then the polynomial expansion

$$\frac{\mathrm{d}\mathbb{P}_\mathbf{X}}{\mathrm{d}\boldsymbol{\nu}}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{N}^n} a_\mathbf{k} Q_\mathbf{k}(\mathbf{x}) = \sum_{k_1=0}^{+\infty} \ldots \sum_{k_n=0}^{+\infty} a_{k_1,\ldots,k_n} Q_{k_1}(x_1) \ldots Q_{k_n}(x_n), \tag{23}$$

is valid and the polynomial approximations of the multivariate **p.d.f.** follows by truncation

$$f_{\mathbf{X}}^{\mathbf{K}}(\mathbf{x}) = \sum_{k_1=0}^{K_1} \dots \sum_{k_n=0}^{K_n} a_{k_1,\dots,k_n} Q_{k_1}(x_1) \dots Q_{k_n}(x_n) f_{\nu_1}(x_1) \dots f_{\nu_n}(x_n), \tag{24}$$

where $\mathbf{K} = (K_1, \dots, K_n)$. The coefficients of the expansion are given by

$$a_{\mathbf{k}} = \mathbb{E}\left[Q_{k_1}(X_1) \dots Q_{k_n}(X_n)\right]. \tag{25}$$

The expression of the coefficients (25) is close to the one-dimensional one (17), however the number of coefficients has drastically increased, which produces a significant increase in the computational effort required. The probability density function approximation derived in (18) and (24) admit a tractable expresion which is valid for every $x$. Note also that the cumulative distribution function and survival function are accessible from (18) and (24) via direct integration.

## 3.2 Application to compound distributions

The goal of this work is to approximate the **p.d.f.** of positive random variable $X = \sum_{i=1}^{N} U_i$ governed by a compound distribution. The probability measure $\mathbb{P}_X$ has support on $\mathbb{R}^+$, so it is natural to choose the gamma distribution as a starting point for the polynomial approximation. The gamma distribution $\Gamma(r, m)$ admits a **p.d.f.** of the form

$$f_\nu(x) = \frac{e^{-\frac{x}{m}} x^{r-1}}{\Gamma(r) m^r}, \text{ for } x \in \mathbb{R}_+,$$

the orthonormal polynomials associated to the gamma distribution are given by

$$Q_k(x) = (-1)^k \left[\frac{\Gamma(n+r)}{\Gamma(n+1)\Gamma(r)}\right]^{-1/2} L_k^{r-1}\left(\frac{x}{m}\right), \text{ for } k \in \mathbb{N}, \tag{26}$$

where $L_k^{r-1}$ are the generalized Laguerre polynomials, defined for instance in the book of Szego [34]. Recall from equation (3) that the probability measure of a compound distribution admits a singular part and a continuous part which is problematic because the absolute continuity of $\mathbb{P}_X$ with respect to $\nu$ is needed. Thus, the singular part is put aside and the polynomial approximation is employed to recover the defective probability density function associated to the continuous part $\mathbb{G}_X$. Therefore, if $\frac{\mathrm{d}\mathbb{G}_X}{\mathrm{d}\nu} \in \mathscr{L}^2(\nu)$ then the polynomial approximation is defined as

$$g_X^K(x) = \sum_{k=0}^{K} a_k Q_k(x) f_\nu(x). \tag{27}$$

Integrating (27) gives access to the distribution function, the survival function and the stop-loss premium. The integrability condition is satisfied as long as

$$g_X(x) = O\left(e^{-\delta x}\right) \text{ as } x \to +\infty \text{ with } \delta > \frac{1}{2m}$$

The condition on the right tail is satisfied if $\gamma_X = \sup\{s > 0 \; ; \; \mathscr{L}_X(s) < +\infty\}$ exists, where $\mathscr{L}_X(s) = \mathbb{E}\left(e^{sX}\right)$ is the moment generating function. If $\gamma_X$ exists then

$$g_X(x) = O\left(e^{-\gamma_X x}\right) \text{ as } x \to +\infty, \tag{28}$$

and the parameters of the reference gamma distribution can be chosen according to

$$r \leq 1 \; ; \; m > \frac{1}{2\gamma_X}.$$

The discussion around the constraints on the shape parameter $r$ is omitted here, but can be found in [11] Chapter 2, Corollary 2. The polynomial approximation is applied to a simpler multivariate compound distribution than the one proposed in (7) as a random vector of the form

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ \vdots \\ V_{nj} \end{pmatrix}, \tag{29}$$

is considered. One has to put aside the singular part of the distribution

$$\mathrm{d}\mathbb{P}_X(\mathbf{x}) = f_N(0) + \mathrm{d}\mathbb{G}_{\mathbf{X}}(\mathbf{x}),$$

where $\delta_{\mathbf{0}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \delta_0(x_i)$ and $\mathbf{0} = (0, \ldots, 0)$, $(\mathbf{V}_j)_{j \in \mathbb{N}}$ is a sequence of **i.i.d.** random vectors. Many multivariate distributions on $\mathbb{R}_+^n$ are listed in [4]. It is also possible to start from given marginal distributions and specify the dependence structure through copulas. The multivariate reference probability measure is defined as a product of gamma measure

$$\boldsymbol{\nu}(\mathbf{x}) = \prod_{i=1}^{n} \nu_i(x_i),$$

where $\nu_i = \Gamma(r_i, m_i)$. The associated multivariate orthonormal polynomials are given by

$$Q_{\mathbf{k}}(\mathbf{x}) = \prod_{i=1}^{n} Q_{\mathbf{k}}^{\nu_{\mathbf{i}}}(x_i),$$

where the polynomials $\{Q_{\mathbf{k}}^{\nu_{\mathbf{i}}}\}$ are defined in (26). The integrability condition is satisfied if

$$g_{\mathbf{X}}(\mathbf{x}) = O\left(e^{-\boldsymbol{\delta}.\mathbf{x}}\right) \text{ as } x_i \to +\infty \text{ for } i = 1, \ldots, n, \text{ with } \boldsymbol{\delta} > \mathbf{m}^{-1}/2, \tag{30}$$

where $\mathbf{m} = (m_1, \ldots, m_n)$ stores the mean parameters of the reference gamma distributions and $\boldsymbol{\delta} > \mathbf{m}^{-1}/2$ means that $\delta_i > \frac{1}{2m_i}$, for $i = 1, \ldots, n$.[16] Condition (30) indicates that the target **p.d.f.** admits an exponential decay in every direction. This condition is satisfied if the multivariate moment generating function $\mathscr{L}_{\mathbf{X}}(\mathbf{s}) = \mathbb{E}\left(e^{\mathbf{sX}}\right)$ is well defined for some $\mathbf{s} \in \mathbb{R}_+^n$. The set of vectors $\Gamma_X = \sup\{\mathbf{s} \in \mathbb{R}_{+*}^n \; ; \; \mathscr{L}_{\mathbf{X}}(\mathbf{s}) < +\infty\}$ is a set of limiting vectors such that if there exists $\boldsymbol{\gamma}_{\mathbf{X}} \in \Gamma_X$, then

$$g_{\mathbf{X}}(x) = O\left(e^{-\boldsymbol{\gamma}_{\mathbf{X}}.\mathbf{x}}\right) \text{ as } x_i \to +\infty \text{ for } i = 1, \ldots, n. \tag{31}$$

[16] A partial ordering on the set of vectors is defined here.

The parameters of the reference distribution are chosen according to

$$\mathbf{r} \geq (1, \ldots, 1) \text{ and } \mathbf{m}^{-1} < 2\boldsymbol{\gamma}_{\mathbf{X}}.$$

Practically, one has to find a vector $\mathbf{s} \in \mathbb{R}^n_{+*}$ such that $\mathscr{L}_{\mathbf{X}}(\mathbf{s}) < +\infty$ and set the mean parameters so that $\frac{1}{m_i} < 2\gamma_{\mathbf{X},i}$, for $i = 1, \ldots, n$, where $\boldsymbol{\gamma}_{\mathbf{X}} = (\gamma_{\mathbf{X},1}, \ldots, \gamma_{\mathbf{X},n})$. The discussion on the choice of $\mathbf{r}$ is again omitted, for more details on these results including proofs the reader is referred to [11].

### 3.3 On the practical evaluation of the coefficients of the expansion

Consider the polynomial representation of the defective **p.d.f.** $g_X(x) = \sum a_n Q_n(x) f_v(x)$, and compute the Laplace transform

$$\begin{aligned} \mathscr{L}_{g_X}(s) &= \int_0^{+\infty} e^{sx} \sum a_n Q_n(x) f_v(x) \mathrm{d}x \\ &= \sum_{k=0}^{+\infty} a_k c_k \left(\frac{1}{1-sm}\right)^r \left(\frac{sm}{1-sm}\right)^k \\ &= \left(\frac{1}{1-sm}\right)^r \mathscr{C}\left(\frac{sm}{1-sm}\right), \end{aligned} \tag{32}$$

where $c_k = \binom{k+r-1}{k}^{-1/2}$ and $z \mapsto \mathscr{C}(z)$ is the generating function of the sequence $(a_k c_k)_{k \in \mathbb{N}}$. The generating function may be expressed in terms of the Laplace transform by inverting (32)

$$\mathscr{C}(z) = (1+z)^{-r} \mathscr{L}_X\left[\frac{z}{m(1+z)}\right], \tag{33}$$

and the coefficients of the polynomial expansion follow from

$$a_k = \frac{1}{k! c_k} \left[\frac{\mathrm{d}^k}{\mathrm{d}z^k} \mathscr{C}(z)\right]_{z=0} \text{ for } k \in \mathbb{N}. \tag{34}$$

Note that the same relationship exists between the generating function of the coefficients of the polynomial representation and the Laplace transform in the multivariate case. The computation are done using Mathematica. The code is available online, see [12].

## 4 A word about the competitors of the polynomial approximation

In the numerical illustrations the performance of the polynomial approximation technique is compared to three different methods: Panjer's algorithm, the exponential moment technique and the direct Fourier transform inversion. A brief description of these numerical methods is proposed here.

## 4.1 Panjer's algorithm

Panjer's algorithm is a recursive method that gives the exact probability of a compound distribution $(\mathbb{P}_N, \mathbb{P}_U)$, when the claim sizes are lattice. The algorithm is based on the existence of a recurrence relationship among the probabilities of the claim frequencies, namely

$$f_N(k) = \left(a + \frac{b}{k}\right) f_N(k-1),\ k = 1, 2, \ldots \tag{35}$$

The only distribution satisfying such a relationship are Poisson, binomial, nd negative binomial distributions. The values of $a$ and $b$ in (35) are determined by the parameters of the underlying probability distribution, see [20, 33]. The probability distribution of $X$ is derived from

$$f_X(k) = \begin{cases} \mathscr{G}_N\left[f_U(0)\right] & \text{For } k = 0, \\ \frac{1}{1 - a f_U(0)} \sum_{j=1}^{k} \left(a + \frac{bj}{k}\right) f_U(j) f_X(k-j) & \text{For } k \geq 1, \end{cases} \tag{36}$$

where $\mathscr{G}_N(s) = \mathbb{E}(s^N)$ denotes the probability generating function of $N$. The recursion(36) is exact when $U$ is a discrete random variable. When it is not the case, the probability distribution of the claim sizes needs to be arithmetized. To do so, a rounding to the closest integer strategy, see [10], is employed. The **p.d.f.** of the claim sizes is approximated by

$$\tilde{f}_U = \begin{cases} F_U\left(\frac{h}{2}\right) & \text{for } k = 0, \\ F_U\left(kh + \frac{h}{2}\right) - F_U\left(kh - \frac{h}{2}\right) & \text{for } k \geq 1. \end{cases} \tag{37}$$

The application of the algorithm (36) yields the approximation

$$f_X(x) \approx \tilde{f}_X(kh),\ k = 0, 1, \ldots.$$

The accuracy increases when the grid width $h$ decreases. The major issue is that a small $h$ entails large computing time especially when approximating the right tail of the distribution.

## 4.2 Exponential moments based technique

Let $Y$ be a continuous random variable, having probability distribution $\mathbb{P}_Y$ on $[0, 1]$. The idea is to approximate the **c.d.f.** of $Y$ by the **c.d.f.** of $\mathscr{B}(n, Y)$ that is an infinite mixture of binomial distributions. The approximation is justified by a convergence result implying that

$$\tilde{F}_Y(y) = \sum_{k=0}^{\lfloor ny \rfloor} \sum_{j=k}^{n} \binom{n}{j} \binom{j}{k} (-1)^{j-k} \mathbb{E}\left(Y^j\right) \to F_Y(y) \text{ as } n \to +\infty.$$

Including more moments in the formula enhances the quality of the approximation. A refinement is needed to get an approximation for $X$ which is a random variable on $\mathbb{R}_+$. The change of variable $Y = X^{-b}$, where $b$ is a scaling parameter leads to the following approximation of $X$'s **c.d.f.**

$$\tilde{F}_X(x) = 1 - \sum_{k=0}^{\lfloor ny \rfloor} \sum_{j=k}^{n} \binom{n}{j} \binom{j}{k} (-1)^{j-k} \mathscr{L}_X\left[-j \ln(b)\right]. \tag{38}$$

It is not quite clear which is the optimal value for $b$, only that it has to be between $1$ and $\exp(1)$. The **p.d.f.** of $X$ can also be derived through differentiation of the approximation (38), namely

$$\tilde{f}_X(x) = \frac{n+1}{n} \frac{\tilde{F}_X(x_i) - \tilde{F}_X(x_{i-1})}{x_i - x_{i-1}}, \; x \in [x_{i-1}, x_i[,$$

where $x_i = \frac{\ln(n) - \ln(n-i+1)}{\ln(b)}$ for $i = 1, \ldots, n$. The final approximation for the **p.d.f.** is given by

$$\tilde{f}_X(x) = \frac{\lfloor nx^{-b} \rfloor \ln(b) \Gamma(n+2)}{n\Gamma(\lfloor nb^{-x} \rfloor + 1)} \sum_{k=0}^{n-\lfloor nb^{-x} \rfloor} \frac{(-1)^k \mathscr{L}_X[-\ln(b)(k + \lfloor k + nb^{-x} \rfloor)]}{k!\Gamma(n - \lfloor nb^{-x} \rfloor - k + 1)} \tag{39}$$

In practice, it has been recommended to evaluate (38) and (39) over the lattice grid $(x_i)_{i=1,\ldots,n}$ and interpolate the values to get a final approximation. The method is described in the work of Mnatsakanov et al. [22, 24, 23, 26].

## 4.3 The direct Fourier transform inversion technique

The direct Fourier transform inversion technique efficiently recovers the survival functions of probability distributions. The approximation is employed as a benchmark when no closed form formula is available. The underlying idea is to start from the integral inversion formula and approximate it through a trapezoidal rule. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. If $g$ is bounded then $\int_{-\infty}^{+\infty} |\mathscr{L}_g(is)| \mathrm{d}s < +\infty$, and

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-isx} \mathscr{L}_g(is) \mathrm{d}s, \; x \in \mathbb{R}. \tag{40}$$

The survival function of $X$ maps $\mathbb{R}_+$ into $[0, 1]$, the function defined as

$$g(x) = \begin{cases} e^{-s_0 x} \overline{F_X}(x) & \text{for } x \geq 0, \\ g(-x) & \text{for } x < 0. \end{cases}$$

where $s_0 > 0$, is a suitable candidate for inversion through formula (40). Actually applying formula (40) leads to an integral representation of the survival function

$$\overline{F_X}(x) = \frac{2e^{s_0 x}}{\pi} \int_0^{+\infty} \cos(xs) \Re \left[ \mathscr{L}_{\overline{F_X}}(-s_0 - is) \right] \mathrm{d}s,$$

which is approximated via a trapezoidal rule

$$\overline{F_X}(x) \approx \frac{2e^{s_0 x}}{\pi} h \left\{ \frac{\mathscr{L}_{\overline{F_X}}(s_0)}{2} + \sum_{k=1}^{+\infty} \cos(xkh) \Re \left[ \mathscr{L}_{\overline{F_X}}(-s_0 - ikh) \right] \right\}, \tag{41}$$

where $h$ is a discretization step. The problem of evaluating the infinite series in (41) is overcome via truncation. The trick is to pick $h$ and $s_0$ so as to control the discretization error on one hand

and to reveal an alternating series that allow for the use of techniques to speed up the partial sums convergence on the other hand. Let $h = \frac{\pi}{2x}$ and $s_0 = \frac{a}{2x}$, (41) becomes

$$\overline{F_X}^a(x) = \frac{e^{a/2}}{2x}\left\{\mathscr{L}_{\overline{F_X}}\left(-\frac{a}{2x}\right) - 2\sum_{k=1}^{+\infty}(-1)^{k+1}\Re\left[\mathscr{L}_{\overline{F}_X}\left(-\frac{a+2\mathrm{i}\pi k}{2x}\right)\right]\right\}. \tag{42}$$

It is possible to show that $|\overline{F_X}(x) - \overline{F_X}^a(x)| < \frac{e^{-a}}{1-e^{-a}}$. The accuracy increases with $a$, however setting $a$ too high generates rounding error that make the approximation numerically unstable. The serie in (42) is of the alternating type and type and the convergence speed can be increased by using EulerâĂŹs summation formula. The final approximation is given by

$$\overline{F_X}^{a,K,M}(x) = \sum_{m=0}^{M}\binom{M}{m}2^{-M}\overline{F_X}^{a,K+m}(x), \tag{43}$$

where

$$\overline{F_X}^{a,K}(x) = \frac{e^{a/2}}{2x}\left\{\mathscr{L}_{\overline{F_X}}\left(-\frac{a}{2x}\right) - 2\sum_{k=1}^{+\infty}(-1)^{k+1}\Re\left[\mathscr{L}_{\overline{F}_X}\left(-\frac{a+2\mathrm{i}\pi k}{2x}\right)\right]\right\}. \tag{44}$$

More information about this procedure can be found in [1].

## 5 Numerical illustrations

The polynomial approximation technique is applied to address the three actuarial science problems presented in Section 2. The accuracy is measured via a relative error criterium defined as

$$\Delta f(x) = \frac{f_{\text{exact}}(x) - f_{\text{approximated}}(x)}{f_{\text{exact}}(x)},$$

where $f$ is the desired function[17], and expressed as a percentage. Note that the exact value of the survival function is not available in almost all examples so the approximation resulting from the Fourier transform inversion technique is used as a substitute.

### 5.1 Compound Poisson distribution

The total claim amounts over a given time period is modeled through

$$X = \sum_{k=1}^{N} U_i,$$

where the number of claims $N$ is $\mathscr{P}(4)$-distributed[18], and two probability measures for the claim sizes are envisaged:

[17] Here the survival function is considered, because it is the most relevant for applications.
[18] $\mathscr{P}(4)$ denotes Poisson distribution with parameter $\lambda$.

MODEL 1 $U$ is $\Gamma(2,2)$-distributed.

MODEL 2 $U$ is $\mathscr{U}(0,8)$-distributed[19].

Regarding the polynomial approximation settings, the order of truncation is $K = 75$ for MODEL 1 and MODEL 2. The shape parameter $r$ is set equal to 1 and the mean parameter is determined by moment matching, namely $m = \mathbb{E}(X)$, for MODEL 1 and MODEL 2. This parametrization ensures the validity of the polynomial expansion on which the approximation is based. The discretization step for Panjer's algorithm is $h = 10^{-2}$. The number of exponential moments is $n = 33$ and the scaling parameter is $b = 1.115$ for the exponential moment based technique. The parameters for the direct inversion of the Fourier transform are $M = 15$, $K = 11$, and $a = 18.5$. Figure 1 displays the error of the different approximation methods with respect to the approximation based on the direct inversion of the Fourier transform for MODEL 1.

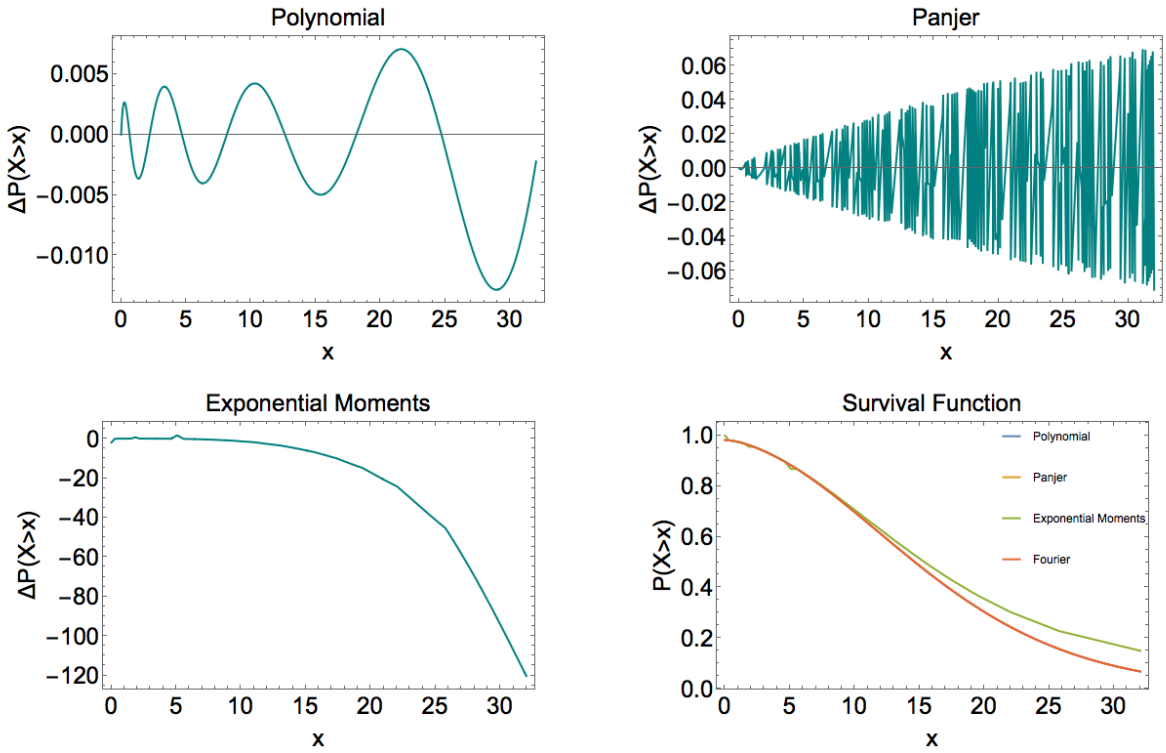


**Figure 1:** Error of the survival function of the compound distribution associated to MODEL 1.

The polynomial approximation outperforms Panjer's algorithm and the exponential moment technique. The exponential moment based method seems to be facing difficulties for large values of $x$. Table 1 displays values of the survival function of the compound distribution associated to MODEL 1 obtained through the different approximations.

[19] $\mathscr{U}(\alpha,\beta)$ denotes the uniform distribution over the set $[\alpha,\beta]$

**Table 1: Survival function values for the compound distribution associated to MODEL 1 via the different approximation methods.**

| $\overline{F_X}(x)$ / $x$ | Polynomial | Fourier | Panjer | Exponential Moments |
|---|---|---|---|---|
| 3.2 | 0.934063 | 0.934099 | 0.93398 | 0.933795 |
| 6.4 | 0.838587 | 0.838553 | 0.838377 | 0.840413 |
| 9.6 | 0.713781 | 0.713808 | 0.713599 | 0.721811 |
| 12.8 | 0.577291 | 0.577288 | 0.577075 | 0.596213 |
| 16. | 0.445215 | 0.445194 | 0.444998 | 0.478344 |
| 19.2 | 0.328711 | 0.328722 | 0.328555 | 0.376255 |
| 22.4 | 0.233307 | 0.233322 | 0.23319 | 0.294725 |
| 25.6 | 0.159783 | 0.159777 | 0.159678 | 0.230831 |
| 28.8 | 0.105919 | 0.105905 | 0.105835 | 0.189891 |
| 32. | 0.0681425 | 0.068141 | 0.0680926 | 0.150105 |

Figure 2 displays the error of the different approximation methods with respect to the approximation based on the direct inversion of the Fourier transform for MODEL $2$. Panjer's algorithm performs better in that case. A greater order of truncation might be needed to reach the same level of accuracy as in MODEL $1$ when using the polynomial approximation.

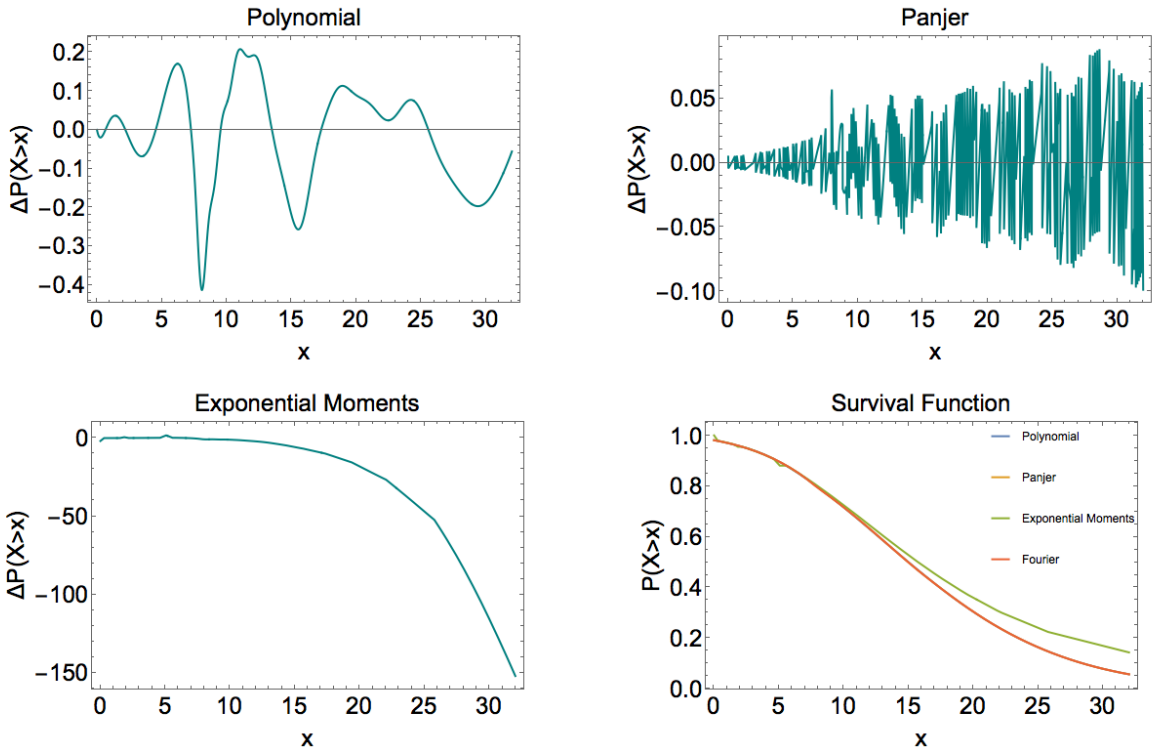


**Figure 2:** Error of the survival function of the compound distribution associated to MODEL 2.

Table 2 shows values of the survival function of the compound distribution associated to MODEL 2 obtained through the different approximations.

**Table 2: Survival function values for the compound distribution associated to MODEL 2 via the different approximation methods.**

| $\overline{F_X}(x)$ / $x$ | Polynomial | Fourier | Panjer | Exponential Moments |
|---|---|---|---|---|
| 3.2 | 0.938961 | 0.938351 | 0.938257 | 0.93827 |
| 6.4 | 0.854291 | 0.855714 | 0.855543 | 0.855596 |
| 9.6 | 0.732761 | 0.73288 | 0.732558 | 0.740374 |
| 12.8 | 0.595608 | 0.596393 | 0.596109 | 0.613427 |
| 16. | 0.457167 | 0.456162 | 0.456011 | 0.490425 |
| 19.2 | 0.33164 | 0.332007 | 0.331828 | 0.382171 |
| 22.4 | 0.228604 | 0.228659 | 0.228532 | 0.295308 |
| 25.6 | 0.150394 | 0.150388 | 0.150301 | 0.227499 |
| 28.8 | 0.0945545 | 0.0943759 | 0.0942929 | 0.184724 |
| 32. | 0.0568335 | 0.0568018 | 0.0567642 | 0.143209 |

The purpose of obtaining accurate approximations of survival functions for these total claim amounts distributions is to permit a comparison of their underlying risk. This can be done by plotting the survival functions on a same graphic. Figure 3 shows the survival function of MODEL 1 and MODEL 2.

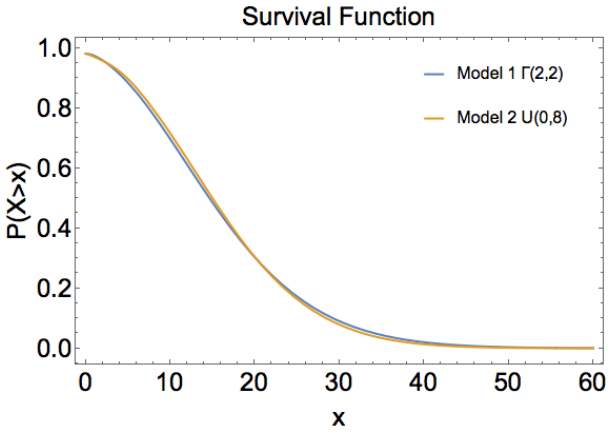


**Figure 3:** Survival function of the compound distribution associated to MODEL 1 and MODEL 2.

The tail is heavier in MODEL $1$ which implies a greater value at risk for a given risk level and therefore a larger risk reserve capital needed to ensure solvability.

In reinsurance, the computation of premium remains a challenging problem numerically. The polynomial approximation is a good operational solution to compute the usual stop-loss premium

$$\Pi_c^{\text{usual}} = \mathbb{E}\left[(X-c)_+\right],$$

linked to a global, non-proportional reinsurance treaty. Figure 4 displays the polynomial approximation of the usual stop-loss premium for MODEL $1$ and $2$, accompanied by a Monte Carlo confidence interval of level $99.9\%$ derived through a Crude Monte Carlo[20] procedure with $10^5$ replications.

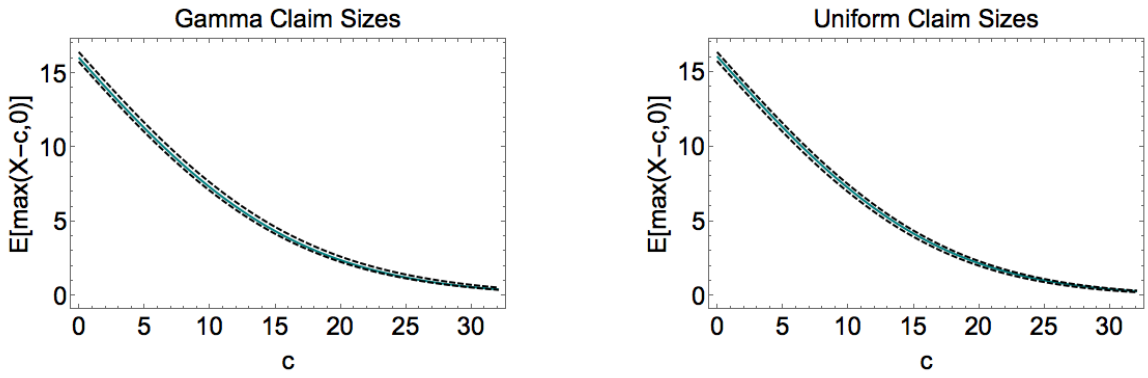


**Figure 4:** Usual stop-loss premium associated to MODEL $1$ and MODEL $2$. (Solid, Dark Green) : Polynomial approximation. (Dashed, Black): Monte Carlo Condidence Set of level $99,9\%$.

A more market consistent reinsurance treaty specifies a limit $d$ for the surplus of the claim amounts over the retention level. The reinsurer's pay off becomes $\min\left[(X-c)_+, d\right]$. This type of contract prevents the insurer from over-estimating his losses and consequently over-charging the reinsurance company. The premium for this reinsurance treaty, called here practical stop-loss premium, is defined as

$$\Pi_{c,d}^{\text{practical}} = \mathbb{E}\left\{\min\left[(X-c)_+, d\right]\right\}.$$

Figure 5 displays a $3D$ visualization of the values of the practical stop-loss premium of the polynomial approximation for MODEL $1$ and MODEL $2$.

A more conventional visualization is provided on Figure 6 where the polynomial approximation of the practical stop-loss premium is plotted for given priorities $c = 16, 32$ accompanied by the classical Monte Carlo confidence interval of level $99.9\%$.

[20] Crude Monte Carlo means standard Monte Carlo method to estimate an expectation.

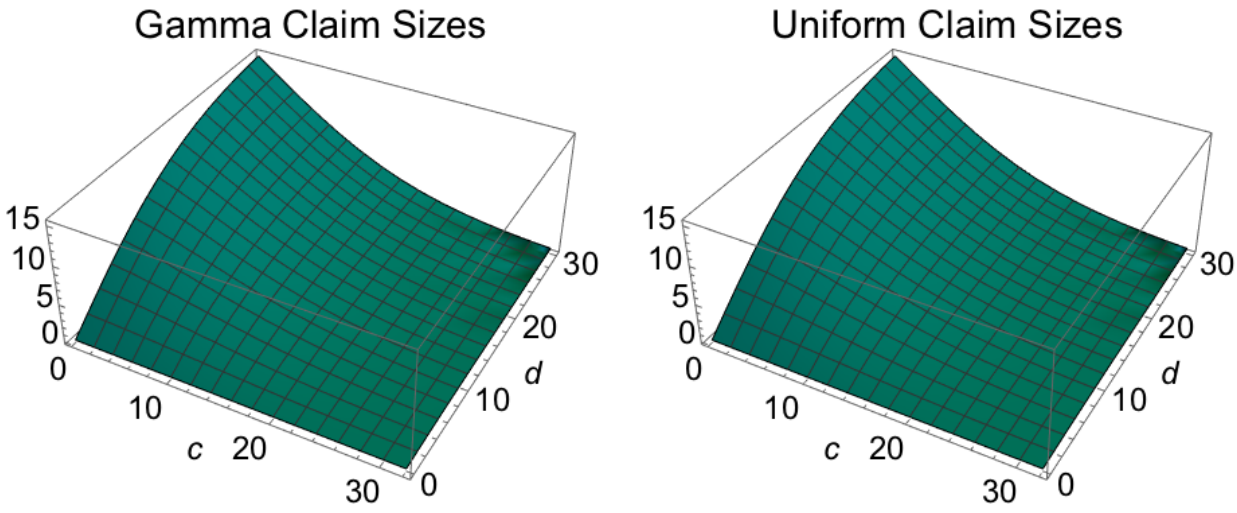


**Figure 5:** Practical stop-loss premium associated to MODEL 1 and MODEL 2.

Once again, MODEL 1 is riskier than MODEL 2 as the stop-loss premiums (both usual and practical) are greater. The operational advantage of the polynomial methodology is that once the coefficients of the expansion has been computed, one has only to plug them into formulas made of very well known and easily computable functions to evaluate interesting quantities such as survival functions and stop-loss premiums in a timely manner.

## 5.2 Bivariate Collective model with application to reinsurance

Consider a reinsurance company that offers a global non-proportional reinsurance treaty with a priority and a limit to two non-life insurance companies that operate on the same market. Denote by $c_1, c_2$ the priorities and $d_1, d_2$ the limits of the reinsurance treaties. The aggregated claim amounts of the two insurers are modeled through a random vector

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \sum_{j=1}^{M} \begin{pmatrix} V_{1j} \\ V_{2j} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^{N_1} U_{1j} \\ \sum_{j=1}^{N_2} U_{2j} \end{pmatrix}, \tag{45}$$

which is basically a two dimensional version of (7), except that independence is assumed in the vector $\left(\sum_{j=1}^{N_1} U_{1j}, \sum_{j=1}^{N_2} U_{2j}\right)$. Indeed, the claim sizes $\{U_{ij}\}_{i,j\in\mathbb{N}}$ form a bivariate sequence of **i.i.d.** random variables independent from the numbers of claims. The claim frequencies $N_1$ and $N_2$ are also mutually independent but not necessarily identically distributed. An indicator of the risk exposure of the reinsurer is the survival function of the random variable

$$Z = \min\left[(X_1 - c_1, d_1\right] + \min\left[(X_2 - c_2, d_2\right], \tag{46}$$

The joint **p.d.f.** of the random vector (45) is approximated through the polynomial method in order to compute the survival function of $Z$. One must carefully remove the different singularities in the

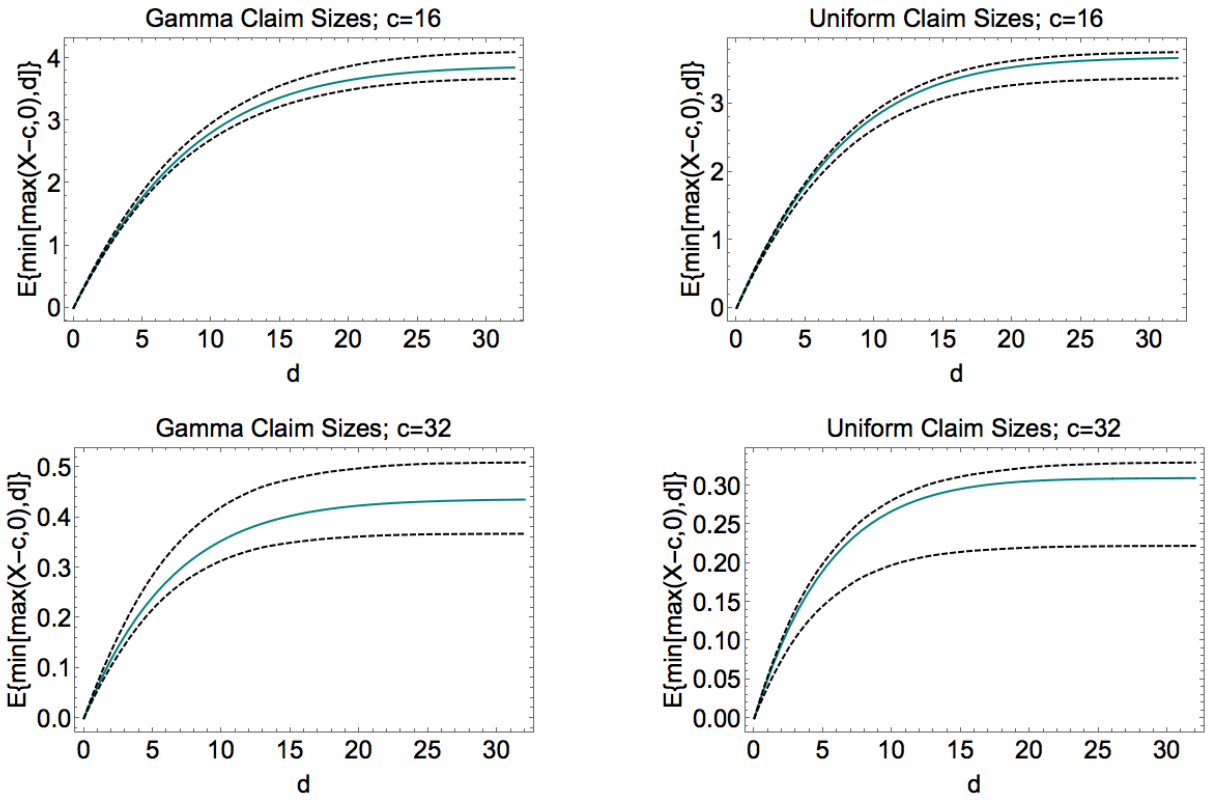


**Figure 6:** Practical stop-loss premium with priorities $c = 16, 32$ associated to MODEL 1 and MODEL 2. (Solid, Dark Green) : Polynomial approximation. (Dashed,Black): Monte Carlo Condidence Set of level $99,9\%$

distribution arising because of the possibility that $M = 0$, $N_1 = 0$ or $N_2 = 0$. Afterward, a bivariate polynomial approximation is derived for the joint **p.d.f.** of $\sum_{j=1}^{M}(V_{1j}, V_{2j})$, followed by two univariate polynomial expansions for $\sum_{j=1}^{N_1} U_{1j}$ and $\sum_{j=1}^{N_2} U_{2j}$. The joint **p.d.f.** of $(X_1, X_2)$ follows from a two dimensional convolution procedure between the polynomial approximations. The survival function is written in terms of an expectation

$$\mathbb{P}(Z > z) = \mathbb{E}\left[\mathbf{1}_{\{\min[(X_1-c_1)_+,d_1]+\min[(X_2-c_2)_+,d_2]>z\}}\right],$$

that is evaluated through brute force numerical integration. The survival function is compared to the one obtained using Crude Monte Carlo. Hereafter the setting of the model (7)

- The claim frequency of the dependent part $M$ is geometrically distributed $\mathcal{NB}(1, 1/2)$
- The claim sizes of the dependent part $\{\mathbf{V}_j\}_{j\in\mathbb{N}}$ are governed by a Downton Bivariate Exponential distribution $DBVE(\mu_1, \mu_2, \rho)$, see [8], where $\mu_1 = \mu_2 = 1$ represents the mean parameters of the marginal distributions, and $\rho = 1/4$ indicates the strength of the correlation among the components.

- The claim frequencies of the independent part $N_1$ and $N_2$ are both geometrically distributed $\mathcal{NB}(1,3/4)$.
- The claim sizes in the independent part $\{U_{ij}\}_{i,j\in\mathbb{N}}$ are exponentially distributed $\Gamma(1,1)$.

In view of the similarity of the total claim amounts distributions, it seems natural to propose the same reinsurance treaty to the insurance companies. In this example, the priorities are set to $c_1 = c_2 = 1$, and the limits are set to $d_1 = d_2 = 4$. Concerning the bivariate polynomial expansion, the shape parameters are $r_1 = r_2 = 1$, the mean parameters are set to $m_1 = \frac{1}{\mu_1(1-\rho)}$ and $m_2 = \frac{1}{\mu_2(1-\rho)}$, and the order of truncation is set to $K = 5$. There is no need to perfom univariate polynomial approximation for the independent part as the **p.d.f.** of geometric compound distribution with exponentially distributed claim amounts is known. Figure 7 displays the polynomial approximation of the survival function of $Z$ along with the Crude Monte Carlo estimator computed with $10^5$ replications and used for verification purposes.

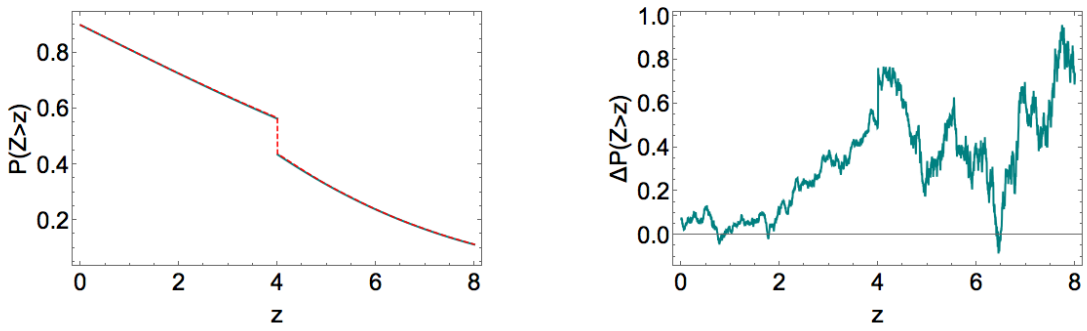


**Figure 7:** Polynomial approximation of the survival function of the reinsurer risk exposure and relative deviation from the Crude Monte Carlo estimation. (Solid, Dark Green) : Polynomial approximation. (Dashed,Red): Monte Carlo approximation.

The results are encouraging as the polynomial approximation is quite close to the Monte Carlo estimation. The recovery of joint probability density functions from the knowledge of their moment generating functions, even in two-dimensional contexts, is known to be a difficult task. The polynomial aproximation shows potential to be employed in real world examples even if at least ten insurers should be taken into account!

## 5.3 Ultimate ruin probability in the compound Poisson ruin model

The financial reserves of a non-life insurance company is modeled through a classical ruin model

$$R_t = u + ct - \sum_{i}^{N_t} U_i,$$

described in Subsection 2.3. Two ruin models are studied, in which the intensity of the Poisson process is $\lambda = 4$ but the claim sizes distribution differ,

MODEL 1 $U$ is $\Gamma(2,2)$-distributed,

MODEL 2 $U$ is $\mathscr{U}(0,8)$-distributed.

Note that the same claim sizes distribution have been considered in Subsection 5.1. The safety loading is equal to $20\%$ in both cases. The shape parameters of the polynomials approximation are set to $1$ and the mean parameter must be equal to $1/\gamma$, where $\gamma$ is the only positive solution to the so-called Cramér-Lundberg equation

$$\lambda\left[\mathscr{L}_U(s)-1\right]-cs=0, \tag{47}$$

that links the moment generating function of the claim sizes distribution to the parameters of the ruin model. This equation has to be solved[21] for the two ruin models, which yields

$$\gamma_{\text{MODEL 1}}\approx 0.0566912 \;\;;\;\; \gamma_{\text{MODEL 2}}\approx 0.0654507.$$

The mean parameters of the polynomial expansion are then set to $m_{\text{MODEL 1}}=\frac{1}{\gamma_{\text{MODEL 1}}}$ and $m_{\text{MODEL 2}}=\frac{1}{\gamma_{\text{MODEL 2}}}$ following the recommendations issued in [14]. The order of truncation is $K=75$ for both approximations. The discretization step is $h=10^{-1}$. It is greater than in Subsection 4.1 because it is necessarry to go further in the right tail of the distribution which induces a considerable computing time if the discretization step is too low. The number of exponential moments is $n=32$, and the scaling parameter is $b=1.059$ in the exponential moments based approximation. Regarding the direct Fourier transform inversion technique, the parameters are set to $M=15$, $K=11$, and $a=18.5$ which is consistent with respect to what is done in Subsection 4.1. Figure 8 displays the error with respect to the direct Fourier transform inversion technique of the different approximation of the ultimate ruin probability in MODEL 1.

The polynomial approximation greatly outperforms Panjer's algorithm and the exponential moments based method. Table 3 gives some values of the ultimate ruin probability in MODEL 1.

[21] Which is done numerically using the build-in function **NSolve** of Mathematica.

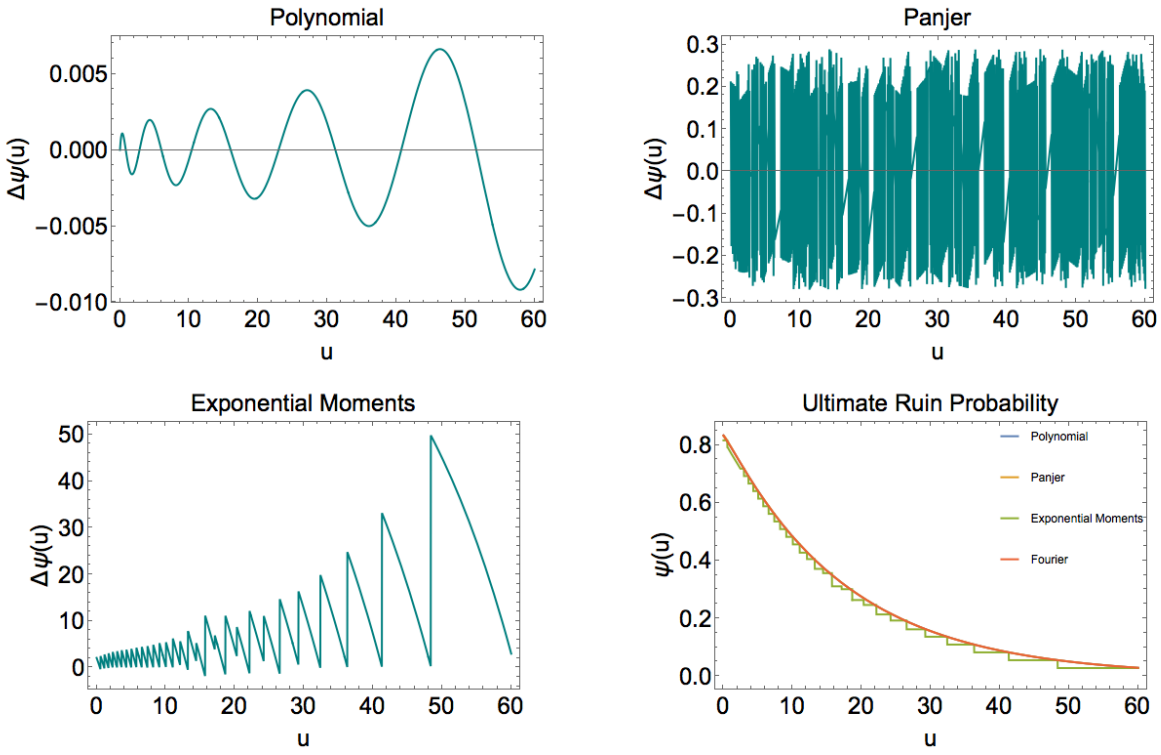


**Figure 8:** Relative error of the approximations of the ultimate ruin probability in MODEL 1.

**Table 3: Ultimate ruin probability values associated to MODEL 1 via the different approximation methods.**

| $\psi(u)$ / $u$ | Polynomial | Fourier | Panjer | Exponential Moments |
|---|---|---|---|---|
| 6. | 0.605967 | 0.605967 | 0.604246 | 0.587937 |
| 12. | 0.431393 | 0.431403 | 0.430169 | 0.427144 |
| 18. | 0.307024 | 0.307016 | 0.306137 | 0.300657 |
| 24. | 0.218489 | 0.218493 | 0.217865 | 0.213342 |
| 30. | 0.155491 | 0.155494 | 0.155046 | 0.136012 |
| 36. | 0.110665 | 0.11066 | 0.11034 | 0.108852 |
| 42. | 0.0787508 | 0.0787526 | 0.078525 | 0.0547796 |
| 48. | 0.0560422 | 0.0560455 | 0.0558832 | 0.0547796 |
| 54. | 0.0398876 | 0.0398856 | 0.0397699 | 0.0275445 |
| 60. | 0.0283874 | 0.0283852 | 0.0283027 | 0.0275445 |

Figure 9 displays the error with respect to the direct Fourier transform inversion technique of the different approximations of the ultimate ruin probability in MODEL $2$.

Again the polynomial approximation outperforms its competitors, however the accuracy is worse than the one achieved for MODEL $1$. A larger truncation order is needed to attain the same level of precision. Table 4 gives some values of the ultimate ruin probability in MODEL $2$.

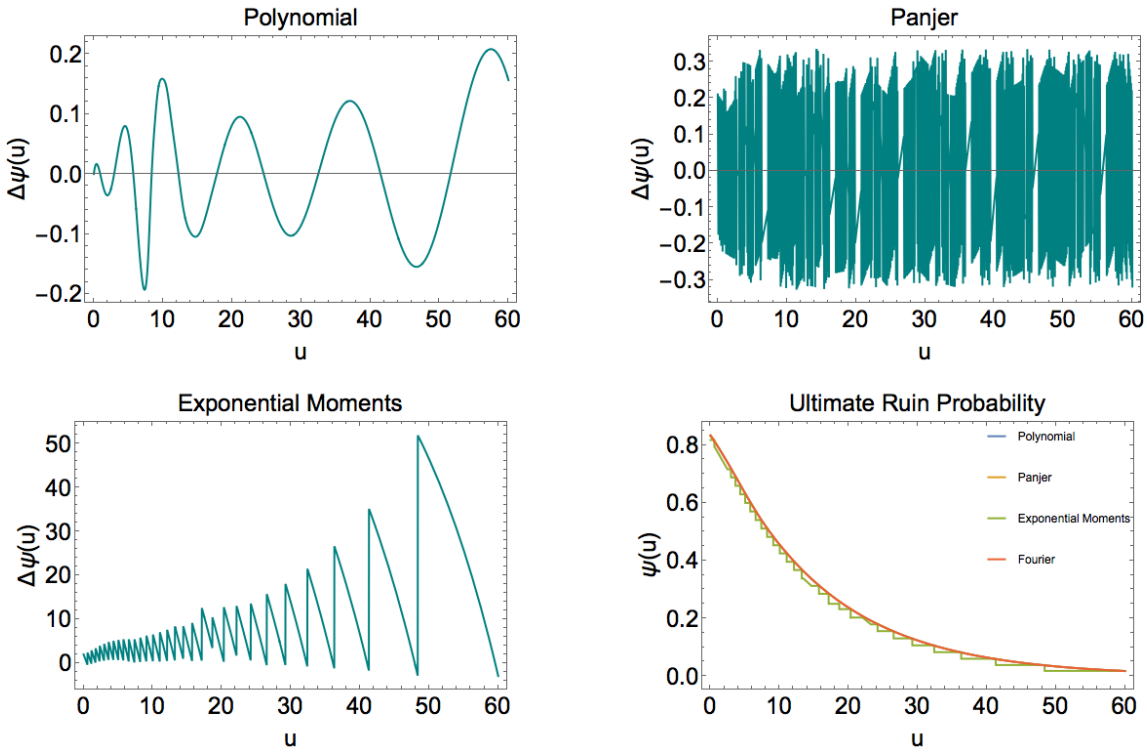


**Figure 9:** Relative error of the approximations of the ultimate ruin probability in MODEL 2.

**Table 4: Ultimate ruin probability values associated to MODEL 2 via the different approximation methods.**

| $\psi(u)$ / $u$ | Polynomial | Fourier | Panjer | Exponential Moments |
|---|---|---|---|---|
| 6. | 0.592845 | 0.592607 | 0.590567 | 0.569688 |
| 12. | 0.399326 | 0.399418 | 0.398085 | 0.395597 |
| 18. | 0.269644 | 0.269669 | 0.268775 | 0.250033 |
| 24. | 0.182041 | 0.182086 | 0.181482 | 0.178938 |
| 30. | 0.123055 | 0.12295 | 0.122541 | 0.106104 |
| 36. | 0.0829246 | 0.0830192 | 0.0827425 | 0.0825649 |
| 42. | 0.0560697 | 0.056057 | 0.0558697 | 0.0380679 |
| 48. | 0.037905 | 0.0378513 | 0.0377245 | 0.0380679 |
| 54. | 0.0255273 | 0.0255583 | 0.0254725 | 0.0177739 |
| 60. | 0.0172307 | 0.0172577 | 0.0171996 | 0.0177739 |

The polynomial approximations of the ultimate ruin probability in MODEL 1 and in MODEL 2 are plotted on Figure 10.

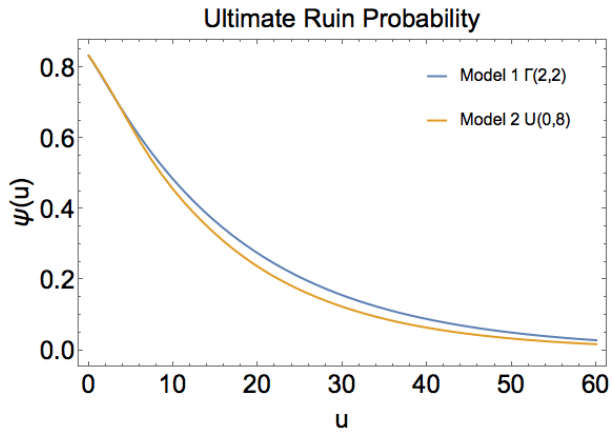


**Figure 10:** Polynomial approximations of the ultimate ruin probability in MODEL 1 and MODEL 2.

The ruin probability is smaller for MODEL 2 which indicates that MODEL 1 is less beneficial for the insurance company than MODEL 2. These results are consitent with the findings of Subsection 4.1. Further numerical results ragarding the approximation of the ultimate ruin probability are available in [14].

## 6 Conclusion

The polynomial aproximation method is efficient and very well suited to study compound distributions. It has been successfully applied to cope with computational difficulties linked to actuarial science problems. The implementation does not require strong coding skills, the numerical illustrations have been carried out using Mathematica. The code is accessible online for the sake of reproducibility[22], see [12]. One of the main drawbacks of the method is the fact that it needs a well defined moment generating function which limits the application to light tailed distributions, however a recent paper [3] shows that the polynomial appoximation can be used to recover the probability density function of the sum of lognormally distributed random variables after a suitabe transformation of the **p.d.f.**. An interesting perspective for further research work is the study of the properties of the statistical estimator of the **p.d.f.** resulting from the replacement of the coefficients of the expansion by their empirical counterparts.

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