EXPONENTIAL CONVERGENCE RATE OF RUIN PROBABILITIES FOR LEVEL-DEPENDENT LÉVY-DRIVEN RISK PROCESSES

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Abstract

We find explicit estimates for exponential rate of long-term convergence for the ruin probability in a level-dependent Lévy-driven risk model, as time goes to infinity. Siegmund duality allows to reduce the problem to long-term convergence of a reflected jump-diffusion to its stationary distribution, which is handled via Lyapunov functions.

Keywords: ruin probability, uniform ergodicity, Lyapunov function, stochastically ordered process, Siegmund duality

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1. Introduction

A non-life insurance company holds at time $t = 0$ an initial capital $u = X(0) \geq 0$, collects premiums at a rate $p(x) > 0$ depending on the current level of the capital $X(t) = x$, and pays from time to time a compensation (when a claim is filed). The aggregated size of claims up to time $t > 0$ is modeled by a compound Poisson process $(L(t)\ , t \geq 0)$. That is, the number of claims is governed by a homogeneous Poisson process of intensity β independent from the claims, which, in turn, form a sequence $U_1, U_2, \ldots$ of i.i.d. nonnegative random variables with cumulative distribution function $B(\cdot)$. The net worth of the insurance company is then given by a continuous-time stochastic process $X = (X(t), t \geq 0)$, with

$$X(t) = u + \int_0^t p(X(s))\mathrm{d}s - \sum_{k=1}^{N(t)} U_k = u + \int_0^t p(X(s))\,\mathrm{d}s - L(t),\ t \geq 0. \tag{1}$$

Examples of such level-dependent premium rate include the insurance company downgrading the premium rate from p_1 to p_2 when the reserves reach a certain threshold; or incorporating a constant interest force: $p(x) = p + ix$. In this work, a more general risk model is considered. The surplus (1) is perturbed by a Brownian motion $\{W(t)\ , t \geq 0\}$, multiplied by a diffusion parameter σ, to account for the fluctuations

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around the premium rate. This diffusion parameter may also depend on $X(t)$. We further let the accumulated liability $L(t)$ be governed by a pure jump, nondecreasing Lévy process, starting from $L(0) = 0$. The financial reserves of the insurance company evolve according to the following dynamics:

$$\mathrm{d}X(t) = p(X(t))\,\mathrm{d}t + \sigma(X(t))\,\mathrm{d}W(t) - \mathrm{d}L(t), \quad X(0) = u. \tag{2}$$

In risk theory, one of the main challenges is the evaluation of ruin probabilities. The probability of ultimate ruin is the probability that the reserves ever drop below zero:

$$\psi(u) := \mathbb{P}\Big(\inf_{t\ge 0} X(t) \le 0\Big). \tag{3}$$

We stress dependence of ψ on the initial capital u. The probability of ruin by time T is defined as

$$\psi(u,T) := \mathbb{P}\Big(\inf_{0\le t\le T} X(t) \le 0\Big). \tag{4}$$

We often refer to $\psi(u)$ and $\psi(u,T)$ as infinite and finite time horizon ruin probability, respectively. For a comprehensive overview on risk theory and ruin probabilities, see the book [3].

We study the rate of exponential convergence of the finite-time horizon ruin probability toward its infinite-time counterpart. The goal of this article is to provide an explicit estimate for such rate: To find constants $C, k > 0$ such that

$$0 \le \psi(u) - \psi(u,T) \le Ce^{-kT}, \quad \text{for all} \quad T, u \ge 0. \tag{5}$$

This is achieved via a duality argument. For the original model (1), define the *storage process* $Y = \{Y(t)\ ,\ t \ge 0\}$ as follows:

$$Y(t) = L(t) - \int_0^t p(Y(s))\,\mathrm{d}s. \tag{6}$$

We assume that $p(y) = 0$ for $y < 0$. This makes zero a reflecting barrier. This is essentially a time-reversed version of the risk model (1), reflected at 0. For the general model (2) perturbed by Brownian motion, the dual process is a reflected jump-diffusion on the positive half-line. As $t \to \infty$, $Y(t)$ weakly converges to some distribution $Y(\infty)$. The crucial observation is: For $T > 0$ and $u \ge 0$,

$$\mathbb{P}(Y(T) \ge u) = \psi(u,T), \quad \mathbb{P}(Y(\infty) \ge u) = \psi(u).$$

This is a particular case of Siegmund duality, see Siegmund [32]. This method was first employed in [17], for the similar duality between absorbed and reflected Brownian motion. It has become a standard tool in risk theory since the seminal paper of Prabhu [25], see also [3, Chapter III, Section 2]. The problem (5) therefore reduces to the study of the convergence of $Y(t)$ toward $Y(\infty)$ as $t \to \infty$:

$$0 \le \mathbb{P}(Y(\infty) > u) - \mathbb{P}(Y(T) \ge u) \le Ce^{-kT}.$$

In this paper in Lemma 6, we state and prove Siegmund duality between the process (1) and a certain reflected diffusion with additional Lévy term. We believe this is a contribution to the theory of Siegmund duality. There exist many similar Siegmund duality results: [10, 15, 16, 33], but we could not find exactly the one we need, so we proved it ourselves.

A *stochastically ordered* real-valued Markov process $Y = \{Y(t)\ , t \ge 0\}$ is such that, for all $y_1 \ge y_2$, we can couple two copies $Y_1(t)$ and $Y_2(t)$ of $Y(t)$ starting from $Y_1(0) = y_1$ and $Y_2(0) = y_2$, in such a way that $Y_1(t) \ge Y_2(t)$ a.s. for all $t \ge 0$. A *Lyapunov function* for a Markov process with generator $\mathcal{L}$ is, roughly speaking, a function $V \ge 1$ such that $\mathcal{L}V(x) \le -cV(x)$ for some constant $c > 0$, for all x outside of a compact set. Then we can combine this coupling method with a Lyapunov function to get a simple, explicit, and in some cases, sharp estimate for the rate k. This method was first applied in Lund and Tweedie [18] for discrete-time Markov chains, and in Lund et al. [19] for continuous-time Markov processes. A direct application of their results yields the rate of convergence for the storage process defined in (6) and the level-dependent compound Poisson risk model (1). However, the dual model associated to the risk process (2) is a more general process since it is a reflected jump-diffusion on the positive half-line.

The same method as in Lund et al. [19] has been refined in a recent paper by Sarantsev [30] and applied to reflected jump-diffusions on the half line. The jump part is not a general Lévy process, but rather a state-dependent compound Poisson process, which makes a.s. finitely many jumps in finite time. In a recent paper [11], it was applied to *Walsh diffusions* (processes which move along the rays emanating from the origin in $\mathbb{R}^d$ as one-dimensional diffusions; as they hit the origin, they choose a new ray randomly). Without attempting to give an exhaustive survey, let us mention classic papers [7, 20, 21] which use Lyapunov functions (without stochastic ordering) to prove the very fact of exponential long-term convergence, and a related paper of Sarantsev [29]. However, the estimation of the rate k is more difficult. Some partial results in this direction are provided in the papers [5, 6, 22, 26, 27, 28].

In this paper, we combine these two methods: Lyapunov functions and stochastic ordering, to find the rate of convergence of the process Y, which is dual to the original process X from (2). This process Y, as noted above, is a reflected jump-diffusion on the half-line. We apply the same method developed in [19, 30]. In the general case, it can have infinitely many jumps during finite time, or can have no diffusion component, as in the level dependent compound Poisson risk model from (1). Therefore, we need to adjust the argument from [30]. Our method only applies in the case of light tailed claim size. Asmussen and Teugels in [4] studied the convergence of ruin probabilities in the compound Poisson risk model with sub-exponentially distributed claim size. It is shown that the convergence takes place at a sub-exponential rate.

The paper is organized as follows. In Section 2, we define assumptions on p, σ, and the Lévy process L. We also introduce the concept of Siegmund duality to reduce the problem to convergence rate of a reflected jump-diffusion to its stationary distribution. Our main results are stated in Section 3: Theorem 3.1 and Corollary 1 provide an estimate for the exponential rate of convergence. Section 4 gives examples of calculations of the rate k. The proof of Theorem 3.1 is carried out in Section 5. Proofs of some technical lemmata are postponed until Appendix.

2. Definitions and Siegmund duality

First, let us impose assumptions on our model (2). Recall that the wealth of the insurance company is modeled by the right-continuous process with left limits $X = (X(t),\, t \ge 0)$, governed by the following integral equation:

$$X(t) = u + \int_0^t p(X(s))\,\mathrm{d}s + \int_0^t \sigma(X(s))\,\mathrm{d}W(s) - L(t), \tag{7}$$

or, equivalently, by the stochastic differential equation (SDE) with initial condition $X(0) = u$, given by (2). We say that X is *driven* by the Brownian motion W and Lévy process L. A function $f : \mathbb{R} \to \mathbb{R}$, or $f : \mathbb{R}_+ \to \mathbb{R}$, is *Lipschitz continuous* (or simply *Lipschitz*) if there exists a constant K such that $|f(x) - f(y)| \le K|x - y|$ for all x, y.

Assumption 1. *The function $p : \mathbb{R}_+ \to \mathbb{R}$ is Lipschitz. The function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is bounded, and continuously differentiable with Lipschitz continuous derivative σ'.*

Assumption 2. *The process L is a pure jump subordinator, that is, a Lévy process (stationary independent increments) with $L(0) = 0$, and with a.s. nondecreasing trajectories, which are right continuous with left limits. The process W is a standard Brownian motion, independent of L.*

Assumption 1 is not too restrictive as it allows to consider classical risk process such as: (a) the compound Poisson risk process when $p(x) = p$, and $\sigma(x) = 0$; (b) the compound Poisson risk process under constant interest force when $p(x) = p + ix$, and $\sigma(x) = 0$. However, the regime-switching premium rate when the surplus hits some target is not covered. Assumption 2 allows the study of the compound Poisson risk process perturbed by a diffusion when $p(x) = p$, and $\sigma(x) = \sigma$, extensively discussed in the paper by Dufresne and Gerber [8], as well as the Lévy-driven risk processes defined for example in Furrer [9] or Morales and Schoutens [24]. It is known from the standard theory, see for example [13, Section 6.2], that the *Lévy measure* of this process is a measure μ on $\mathbb{R}_+$ which satisfies

$$\int_0^\infty (1 \wedge x)\,\mu(\mathrm{d}x) < \infty. \tag{8}$$

Therefore, for all $c > 0$ we have:

$$\mu[c, \infty) < \infty, \text{ and } \int_0^c x\,\mu(\mathrm{d}x) < \infty.$$

When $\mu(\mathbb{R}_+) = \infty$, there are infinitely many jumps on any finite time interval. If $\mu(\mathbb{R}_+) < \infty$: the Lévy measure is finite, then there are finitely many jumps in a finite time interval, and we simply have a compound Poisson process: Times between jumps form i.i.d. exponential random variables with rates $\mu(\mathbb{R}_+)$, and the displacement during each jump is distributed according to the normalized measure $[\mu(\mathbb{R}_+)]^{-1}\,\mu(\cdot)$. From Assumption 2, we have:

$$\mathbb{E}e^{-\lambda L(t)} = \exp\left(t\kappa(-\lambda)\right), \quad \text{for every} \quad t, \lambda \ge 0, \tag{9}$$

where $\kappa(\lambda)$ is the *Lévy exponent:*

$$\kappa(\lambda) := \int_0^\infty \left[e^{\lambda x} - 1\right] \mu(\mathrm{d}x),\ \ \lambda \in \mathbb{R}. \tag{10}$$

As shown in [1, Theorem 3.3.15], applied to the case $b = 0$ and $X(t) \equiv t$ (in the notation of that book), under Assumption 2, L is a Feller continuous strong Markov process, with generator

$$\mathcal{N} f(x) = \int_0^\infty \left[f(x+y) - f(x)\right] \mu(\mathrm{d}y), \tag{11}$$

for $f \in C^\infty(\mathbb{R})$ with a compact support. We impose an additional assumption.

Assumption 3. *The measure μ has finite exponential moment: for some $\lambda_0 > 0$,*

$$\int_1^\infty e^{\lambda_0 x}\, \mu(\mathrm{d}x) < \infty. \tag{12}$$

Remark 1. The existence of exponential moments on the jump size distribution prevents us from considering heavy tailed claim size distribution, see [31, Chapter 5]. This case is treated in the work of Asmussen and Teugels [4] or Sato [31].

Under Assumption 3, we can combine (8) and (12) to get:

$$\kappa(\lambda) < \infty \ \text{ for } \ \lambda \in [0, \lambda_0).$$

The proof of the following technical lemmas are postponed to Appendices A and B.

Lemma 1. *Under Assumption* 3*, we can extend the formula* (11) *for functions $f \in C^\infty(\mathbb{R})$ which satisfy*

$$C_f := \sup_{x \ge 0} e^{-\lambda x} |f(x)| < \infty \textit{ for some } \lambda \in (0, \lambda_0). \tag{13}$$

Lemma 2. *Under Assumptions* 2 *and* 3*, the following quantity is finite:*

$$m(\mu) := \int_0^\infty x\, \mu(\mathrm{d}x) < \infty. \tag{14}$$

Example 2.1. If $\{L(t),\, t \ge 0\}$ is a compound Poisson process with jump intensity β and distribution B for each jump, then the Lévy measure is given by $\mu(\cdot) = \beta B(\cdot)$.

The following lemma can be proved by a classic argument, a version of which can be found in any textbook on stochastic analysis, see for example [13, Section 5.2] or [31, Chapter 6]. For the sake of completeness, we give the proof in the Appendix 3.

Lemma 3. *Under Assumptions* 1 *and* 2*, for every initial condition $X(0) = u$ there exists (in the strong sense, that is, on a given probability space) a pathwise unique version of* (2)*, driven by the given Brownian motion W and Lévy process L. This is* *a Markov process, with generator*

$$\mathcal{L}f(x) := p(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) + \int_0^\infty [f(x-y) - f(x)]\,\mu(\mathrm{d}y) \qquad (15)$$

for $f \in C^2(\mathbb{R})$ with a compact support. Under Assumption 3, this expression (15) is also valid for functions $f \in C^2(\mathbb{R})$ satisfying (13) with $f(-x)$ instead of $f(x)$.

Define the *ruin probability* in finite and infinite time horizons as in (4) and (3). We are interested in finding an estimate of the form

$$0 \le \psi(u) - \psi(u,T) \le Ce^{-kT},\ u, T \ge 0,$$

for some constants C, $k > 0$. Recall the concept of *Siegmund duality.*

Definition 1. Two Markov processes $X = (X(t),\, t \ge 0)$ and $Y = (Y(t),\, t \ge 0)$ on $\mathbb{R}_+$ are called *Siegmund dual* if for all $t, x, y \ge 0$,

$$\mathbb{P}_x(X(t) \ge y) = \mathbb{P}_y(Y(t) \le x).$$

Here, the indices x and y refer to initial conditions $X(0) = x$ and $Y(0) = y$.

Siegmund duality allows us to reduce the ruin problem to a convergence problem of a reflected jump-diffusion $Y = \{Y(t)\ ,\ t \ge 0\}$ toward stationarity. There is a vast literature on Siegmund duality and a more general concept of functional duality of stochastic processes, both in discrete and continuous time. The earliest example of Siegmund duality was [17, p.210]: absorbed and reflected Brownian motions on $\mathbb{R}_+$ are Siegmund dual. In a more general case, duality between absorbed and reflected processes was noted in [32]. Siegmund duality was studied for diffusions, jump-diffusions, and their absorbed and reflected versions, in [15, 16, 33], and for continuous-time discrete-space Markov chains in [37]. The paper [34] deals with Siegmund duality for general partially ordered spaces. See also survey [10] and references therein.

Take some functions p_*, $\sigma_* : \mathbb{R}_+ \to \mathbb{R}$.

Definition 2. Consider an $\mathbb{R}_+$-valued process $Y = (Y(t),\, t \ge 0)$ with right-continuous trajectories with left limits, which satisfies the following SDE:

$$Y(t) = Y(0) + \int_0^t p_*(Y(s))\,\mathrm{d}s + \int_0^t \sigma_*(Y(s))\,\mathrm{d}W(s) + L(t) + R(t), \qquad (16)$$

where $R = (R(t),\, t \ge 0)$ is a nondecreasing right-continuous process with left limits, which starts from $R(0) = 0$ and can increase only when $Y(t) = 0$. Then the process Y is called a *reflected jump-diffusion on the half-line*, with *drift coefficient* p_*, *diffusion coefficient* σ_*, and *driving jump process* L with *Lévy measure* μ.

The following result is the counterpart of Lemma 3 for the process $Y = \{Y(t)\ ,\ t \ge 0\}$.

Lemma 4. *If p_* and σ_* are Lipschitz, then for every initial condition $Y(0) = y$, there exists in the strong sense a pathwise unique version of (16). This is a Markov process* *with generator $\mathcal{A}$, such that for $f \in C^2(\mathbb{R}_+)$ with compact support and $f'(0) = 0$:*

$$\mathcal{A}f(x) = p_*(x)f'(x) + \frac{1}{2}\sigma_*^2(x)f''(x) + \int_0^\infty [f(x+y) - f(x)]\, \mu(\mathrm{d}y). \tag{17}$$

The proof, which is similar to that of Lemma 3, is provided in the Appendix D. It was shown in [32] that a Markov process on $\mathbb{R}_+$ has a (Siegmund) dual process if and only if it is stochastically ordered.

Theorem 1. *A Markov process X, corresponding to a transition semigroup $(P^t)_{t\geq 0}$, is stochastically ordered, if and only if one of the following two conditions holds:*

- *(a) the semigroup $(P^t)_{t\geq 0}$ maps bounded nondecreasing functions into bounded nondecreasing functions; that is, for every bounded nondecreasing $f : \mathbb{R}_+ \to \mathbb{R}$ and every $t \geq 0$, the function $P^t f$ is also bounded and nondecreasing;*
- *(b) for every $t \geq 0$, $c \geq 0$, the function $x \mapsto \mathbb{P}_x(X(t) \geq c)$ is nondecreasing in x.*

This equivalence follows from [12]. Now, consider the process (2), stopped at hitting 0. The following result is known in the literature. A proof is provided in Appendix E for the sake of completeness.

Lemma 5. *The process* (2) *is stochastically ordered.*

Our main convergence theorem relies on the following result. We did not find this exact result in the literature, a proof is given in Appendix F.

Lemma 6. *Under Assumptions* 1 *and* 2*, the Siegmund dual process for the jump-diffusion* (2)*, absorbed at zero, is the reflected jump-diffusion on $\mathbb{R}_+$ from* (16)*, starting at $Y(0) = 0$, with drift and diffusion coefficients*

$$p_*(x) = -p(x) + \sigma(x)\sigma'(x), \quad \sigma_*(x) = \sigma(x). \tag{18}$$

To summarize this section: We have shown that under Assumptions 1, 2, 3, the wealth process is a stochastically ordered Markov process that admits as a Siegmund dual process a Markov process defined as a reflected jump-diffusion process. Therefore, the rate of convergence for ruin probabilities coincides with that of the dual process $Y = \{Y(t)\ ,\ t \geq 0\}$, associated to the risk process $X = \{X(t)\ ,\ t \geq 0\}$, toward stationarity.

3. Main results

A common method to prove an exponential rate of convergence toward the stationary distribution is to construct a Lyapunov function.

Definition 3. Let $V : \mathbb{R}_+ \to [1, \infty)$ be a continuous function and assume there exists $b, k, z > 0$ such that

$$\mathcal{A}V(x) \leq -kV(x) + b1_{[0,z]}(x),\ x \in \mathbb{R}_+. \tag{19}$$

then V is called a *Lyapunov function.*

We shall build a Lyapunov function for the Markov process Y in the form $V_\lambda(x) = e^{\lambda x}$, for $\lambda > 0$. This choice appears to be suitable to tackle the rate of convergence problem of reflected jump-diffusions process as the generator acts on it in a simple way. Under Assumption 3, consider the function

$$\varphi(\lambda, x) := p_*(x)\lambda + \frac{1}{2}\sigma^2(x)\lambda^2 + \kappa(\lambda),\ \lambda \in [0, \lambda_0),\ x \in \mathbb{R}.$$

For a signed measure ν on $\mathbb{R}_+$ and a function $f : \mathbb{R}_+ \to \mathbb{R}$, we denote by $(\nu, f) = \int f \mathrm{d}\nu$. Additionally, for a function $f : \mathbb{R}_+ \to [1, +\infty)$, define the following norm: $\|\nu\|_f := \sup_{|g| \le f} |(\nu, g)|$. If $f \equiv 1$, then $\|\cdot\|_f$ is the total variation norm. Define

$$\Phi(\lambda) = \inf_{x \ge 0} (-\varphi(\lambda, x)) = -\sup_{x \ge 0} \varphi(\lambda, x). \tag{20}$$

Theorem 3.1. *Under Assumptions* 1, 2, 3, *suppose*

$$\Phi(\lambda) > 0 \textit{ for some } \lambda \in (0, \lambda_0). \tag{21}$$

Then there exists a unique stationary distribution π for the reflected jump-diffusion Y. Take a $\lambda \in (0, \lambda_0)$ such that $k = \Phi(\lambda) > 0$. This stationary distribution satisfies $(\pi, V_\lambda) < \infty$. The transition function $Q^t(x, \cdot)$ of the process Y satisfies

$$\|Q^t(x, \cdot) - \pi(\cdot)\|_{V_\lambda} \le [V_\lambda(x) + (\pi, V_\lambda)]\, e^{-kt}. \tag{22}$$

The proof of Theorem 3.1 is postponed until Section 5. The central result of this paper is a corollary of Theorem 3.1, direct consequence of the duality link established between the processes X and Y.

Corollary 1. *Under Assumptions* 1, 2, 3, *and the condition* (21),

$$0 \le \psi(u) - \psi(u, T) \le [1 + (\pi, V_\lambda)]\, e^{-kT}, \quad u, T \ge 0. \tag{23}$$

Proof. In virtue of Siegmund duality,

$$\psi(u) - \psi(u, T) = \mathbb{P}(Y(\infty) \ge u) - \mathbb{P}(Y(T) \ge u), \tag{24}$$

where $Y = (Y(t)\ , t \ge 0)$ is a reflected jump-diffusion on $\mathbb{R}_+$, starting at $Y(0) = 0$, and $Y(\infty)$ is a random variable distributed as π. We may rewrite (24) as $\psi(u) - \psi(u, T) = \pi([u, \infty)) - Q^T(0, [u, \infty))$. Then the inequality (23) follows immediately from the application of Theorem 3.1. $\square$

In the space-homogeneous case: $p(x) \equiv p$ and $\sigma(x) \equiv \sigma$, the quantity $\varphi(\lambda, x)$ is independent of x, and condition (21) means that there exists a $\lambda > 0$ such that $\varphi(\lambda) < 0$. Then $p_* = p$, and $\varphi'(0) = -p + \psi'(0) = -p + m(\mu)$. It is easy to show that $\varphi(\cdot)$ is a convex function with $\varphi(0) = 0$. Therefore, condition (21) holds if and only if $\varphi'(0) < 0$, or, equivalently,

$$p > m(\mu). \tag{25}$$

4. Explicit rate of exponential convergence calculation

In this section, we aim at studying the rate k of exponential convergence depending on the parameters of the risk model. In the examples, we assume a constant premium rate that satisfies the *net benefit condition* with

$$p = (1+\eta) \cdot \mathbb{E}[L(1)], \ \eta > 0,$$

and a constant diffusion parameter σ around the premium rate. Under these settings and for finite measure ν (when L is the compound Poisson process), the rate of exponential convergence, derived in this work, has been shown to be optimal in a certain sense: See the paper [30, Section 6]. However, we absolutely do not claim that for general (non-constant) premium rate p and diffusion parameter σ, this exponential rate is optimal. Let us remark that the concept of optimal rate of convergence could be understood in various ways; at the very least, it depends on the distance used.

4.1. Compound Poisson risk model perturbed by a diffusion

In this subsection, the risk process $X = (X(t)\ , t \geq 0)$ is defined as

$$X(t) = u + pt + \sigma W(t) - \sum_{k=1}^{N(t)} U_k, \tag{26}$$

where $u \geq 0$ denotes the initial capital and p corresponds to the premium rate. The process $W = (W(t)\ , t \geq 0)$ is a standard Brownian motion allowing to capture the volatility around the premium rate encapsulated in the parameter $\sigma > 0$. The process $N = (N(t)\ , t \geq 0)$ is a homogeneous Poisson process with intensity $\beta > 0$, independent from the claims $U_1, U_2, \ldots$ i.i.d. with distribution function B. The premium rate satisfies the *net benefit condition:* $p = (1+\eta)\beta\mathbb{E}(U)$, where $\eta > 0$ is *safety loading.*

We can study the rate of exponential convergence of ruin probabilities; specifically, how it depends on the parameters of the model: (a) the diffusion coefficient σ in front of the perturbation term; (b) the safety loading η; (c) the shape of the claim size distribution. The function $\varphi(\lambda, x)$ for this risk process is given by

$$\varphi(\lambda, x) = -p\lambda + \frac{1}{2}\sigma^2\lambda^2 + \beta\left[\widehat{B}(\lambda) - 1\right], \ \lambda \geq 0, \ x \in \mathbb{R},$$

where $\widehat{B}(\lambda) = \mathbb{E}(e^{\lambda U})$ denotes the moment generating function (MGF) of the claim amount distribution. As the expression of $\varphi(\lambda, x)$ actually does not depend on x then

$$\inf_{x \geq 0}(-\varphi(\lambda, x)) = \Phi(\lambda) = p\lambda - \frac{1}{2}\sigma^2\lambda^2 - \beta\left[\widehat{B}(\lambda) - 1\right], \ \lambda \geq 0, \ x \in \mathbb{R}.$$

Based on Corollary 23, we define the *rate of exponential convergence* by

$$k = \max\{\Phi(\Lambda) \mid \lambda \geq 0\ ;\ \widehat{B}(\lambda) < \infty\}.$$

The function $\lambda \mapsto \Phi(\lambda)$ is strictly concave as $\Phi''(\lambda) = -\sigma^2 - \beta\widehat{B}''(\lambda) < 0$ for all

$\lambda \in \{\lambda \geq 0 \ ; \ \widehat{B}(\lambda) < \infty\}$. It follows that

$$\lambda_* := \underset{\{\lambda \geq 0 \ ; \ \widehat{B}(\lambda) < \infty\}}{\operatorname{argmax}} \Phi(\lambda) \tag{27}$$

is a solution of the equation $p - \sigma^2 \lambda - \beta \widehat{B}'(\lambda) = 0$ under the constraint $\lambda^* \in \{\lambda \geq 0 \ ; \ \widehat{B}(\lambda) < \infty\}$. The rate of exponential convergence is then given by

$$k = \Phi(\lambda_*) = p\lambda_* - \frac{1}{2}\sigma^2 \lambda_*^2 - \beta \left[\widehat{B}(\lambda_*) - 1\right].$$

In this example, we compare the rate of convergence k for three claim size distribution: the *Gamma distribution* $\mathrm{Gamma}(\alpha, \delta)$ with probability density function

$$p(x; \alpha, \beta) = \frac{\delta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\delta x}, \text{ for } x > 0.$$

Define the *exponential distribution* $\mathrm{Exp}(\delta) = \mathrm{Gamma}(1, \delta)$, and the mixture of exponential distributions $\mathrm{MExp}(p, \delta_1, \delta_2)$ with associated probability density function

$$p(x; p, \delta_1, \delta_2) = p\delta_1 e^{-\delta_1 x} + (1 - p)\delta_2 e^{-\delta_2 x}, \ x > 0.$$

Let the claim size be distributed as $\mathrm{Gamma}(2, 1)$. Table 1 gives the rate of exponential convergence for various combinations of values for the safety loading and the volatility. For a given value of the safety loading, the rate of convergences decreases when the

	Safety loading					
Volatility	$\eta = 0.05$	$\eta = 0.1$	$\eta = 0.15$	$\eta = 0.2$	$\eta = 0.25$	$\eta = 0.3$
$\sigma =$ 0	0.00082	0.00319	0.00704	0.01227	0.01881	0.02658
1	0.0007	0.00277	0.00613	0.01073	0.01653	0.02345
2	0.0005	0.00197	0.00439	0.00775	0.01201	0.01716
3	0.00033	0.00132	0.00297	0.00526	0.00819	0.01174
4	0.00023	0.00091	0.00204	0.00361	0.00563	0.0081
5	0.00016	0.00064	0.00145	0.00257	0.00402	0.00578
6	0.00012	0.00048	0.00107	0.0019	0.00297	0.00427
7	0.00009	0.00036	0.00082	0.00145	0.00227	0.00327
8	0.00007	0.00029	0.00064	0.00114	0.00178	0.00257
9	0.00006	0.00023	0.00052	0.00092	0.00144	0.00207
10	0.00005	0.00019	0.00042	0.00075	0.00118	0.0017

TABLE 1: Rate of exponential convergence in the compound Poisson risk model perturbed by a diffusion, with $\mathrm{Gamma}(2, 1)$ claim size.

volatility increases. Conversely, for a given volatility level, the rate of convergence increases with the safety loading. The first row of Table 1 contains the rates of convergence when $\sigma = 0$, associated to the compound Poisson risk model. Figure 1 displays the rates of exponential convergence depending on the volatility level for different values of the safety loading: $\eta = 0.1, 0.2, 0.3$.

Remark 4.1. Consider the compound Poisson risk model perturbed by a diffusion under constant interest force $i > 0$ by assuming that $p(x) = p + ix$, the function

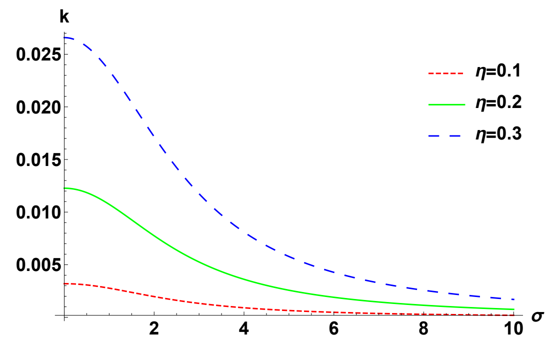


FIGURE 1: The rate of exponential convergence in the compound Poisson risk model perturbed by a diffusion depending on the volatility, for $\eta = 0.1, 0.2, 0.3$.

$\varphi(\lambda, x)$ then becomes

$$\varphi(\lambda, x) = -(p + ix)\lambda + \frac{1}{2}\sigma^2\lambda^2 + \beta\left[\widehat{B}(\lambda) - 1\right], \ \lambda \geq 0, \ x \in \mathbb{R}.$$

Although the function $\varphi(\lambda, x)$ depends on x, it is easily seen that

$$\inf_{x\geq 0}(-\varphi(\lambda, x)) = \Phi(\lambda) = p\lambda - \frac{1}{2}\sigma^2\lambda^2 - \beta\left[\widehat{B}(\lambda) - 1\right], \ \lambda \geq 0, \ x \in \mathbb{R}.$$

The maximization problem is the same as for the compound Poisson risk model perturbed by a diffusion and will lead to the same rate of convergence.

Let us turn to the study of rate of convergence for different claim size distributions. We assume that the claims are either exponentially distributed Exp$(1/2)$, gamma distributed Gamma$(2, 1)$, or mixture of exponential distributed MExp$(1/4, 3/4, 1/4, 3/4)$. The mean associated with claim size distributions is the same, but the variance differs:

$$\mathrm{Var}\left[\mathrm{Gamma}(2,1)\right] < \mathrm{Var}\left[\mathrm{Exp}(1/2)\right] < \mathrm{Var}\left[\mathrm{MExp}(3/4, 3/4, 1/4)\right].$$

Table 2 contains the values of the rate of exponential convergence over the three claim size distributions. The fastest convergence occurs in the gamma case. The slowest convergence occurs in the exponential-mixture case. Figure 2 displays the evolution of the rate of exponential convergence depending on the safety loading and the diffusion parameter for the different assumption over the claim size. In the wake of this numerical study, we may conclude that the speed of convergence depends on the variance of the process. Increasing the variance through the claim size distribution or via the diffusion component makes convergence slower.

4.2. Lévy driven risk process

In this subsection, we compare the rate of exponential convergence of the ruin probabilities when the liability of the insurance company is modeled by a *gamma process* and an *inverse Gaussian Lévy process*. The Lévy measure of a *gamma process*,

			Claim Size Distributions		
Volatility	Safety Loadings		Exp(1/2)	Gamma(2, 1)	MExp(3/4, 3/4, 1/4)
$\sigma = 0$	$\eta =$	0.1	0.00238	0.00319	0.00177
		0.2	0.00911	0.01227	0.00668
		0.3	0.01965	0.02658	0.01426
$\sigma = 1$	$\eta =$	0.1	0.00214	0.00277	0.00163
		0.2	0.00824	0.01073	0.00621
		0.3	0.01791	0.02345	0.01335
$\sigma = 2$	$\eta =$	0.1	0.00163	0.00197	0.00132
		0.2	0.00638	0.00775	0.00511
		0.3	0.01405	0.01716	0.01114
$\sigma = 3$	$\eta =$	0.1	0.00116	0.00132	0.001
		0.2	0.0046	0.00526	0.00392
		0.3	0.01024	0.01174	0.00865
$\sigma = 4$	$\eta =$	0.1	0.00083	0.00091	0.00074
		0.2	0.0033	0.00361	0.00294
		0.3	0.00737	0.0081	0.00654
$\sigma = 5$	$\eta =$	0.1	0.0006	0.00064	0.00056
		0.2	0.00241	0.00257	0.00222
		0.3	0.00541	0.00578	0.00496
$\sigma = 6$	$\eta =$	0.1	0.00045	0.00048	0.00043
		0.2	0.00181	0.0019	0.0017
		0.3	0.00407	0.00427	0.00382
$\sigma = 7$	$\eta =$	0.1	0.00035	0.00036	0.00033
		0.2	0.0014	0.00145	0.00134
		0.3	0.00315	0.00327	0.003
$\sigma = 8$	$\eta =$	0.1	0.00028	0.00029	0.00027
		0.2	0.00111	0.00114	0.00107
		0.3	0.0025	0.00257	0.0024
$\sigma = 9$	$\eta =$	0.1	0.00022	0.00023	0.00022
		0.2	0.0009	0.00092	0.00087
		0.3	0.00202	0.00207	0.00196
$\sigma = 10$	$\eta =$	0.1	0.00019	0.00019	0.00018
		0.2	0.00074	0.00075	0.00072
		0.3	0.00167	0.0017	0.00162

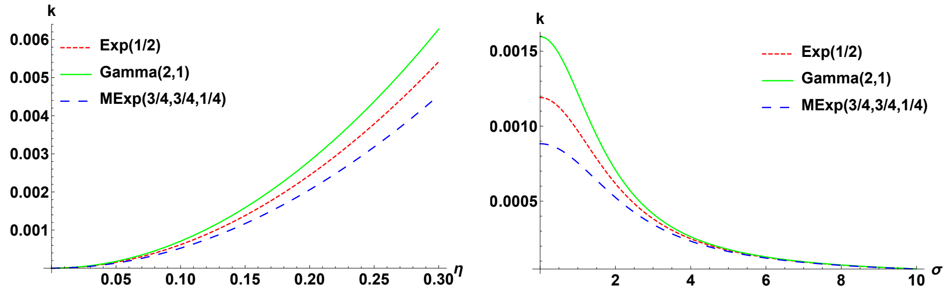
Table 2: Rate of exponential convergence in the compound Poisson risk model perturbed by a diffusion for different claim size distribution.

GammaP(α, β), is given by

$$\mu(\mathrm{d}x) = \frac{\alpha e^{-\beta x}}{x}, \text{ for } x > 0, \tag{28}$$

where α, $\beta > 0$. Its Lévy exponent is

$$\kappa(\lambda) = \alpha \ln\left(\frac{\beta}{\beta - \lambda}\right), \text{ for } \lambda \in [0, \beta). \tag{29}$$



(a) The rate of exponential convergence depending on the safety loading and diffusion $\sigma = 2$.

(b) The rate of exponential convergence depending on the volatility and safety loading $\eta = 0.1$.

FIGURE 2: The rate of exponential convergence in the compound Poisson risk model perturbed by a diffusion for different claim size distributions

Note that in this case, an explicit expression for finite-time and infinite-time ruin probabilities can be found in [22]. The function $\Phi(\cdot)$ is strictly concave as

$$\Phi''(\lambda) = -\sigma^2 - \frac{\alpha}{(\beta - \lambda)^2} < 0.$$

It follows that λ_* is the solution of the equation

$$p - \sigma^2 \lambda - \frac{\alpha}{\beta - \lambda} = 0.$$

The rate of exponential convergence is then given by

$$k = \Phi(\lambda_*) = p\lambda_* - \frac{1}{2}\sigma^2\lambda_*^2 - \alpha \ln\left(\frac{\beta}{\beta - \lambda_*}\right).$$

The Lévy measure of the *inverse Gaussian Lévy process*, IGP(γ), is defined as

$$\mu(\mathrm{d}x) = \frac{1}{\sqrt{2\pi} x^{3/2}} e^{-x\gamma^2/2}, \text{ for } x > 0. \tag{30}$$

where $\gamma > 0$. Its Lévy exponent is

$$\kappa(\lambda) = \gamma - \sqrt{\gamma^2 - 2\lambda}, \text{ for } \lambda \in [0, \gamma^2/2). \tag{31}$$

The function Φ is strictly concave as $\Phi''(\lambda) = -\sigma^2 - (\gamma^2 - 2\lambda)^{-3/2} < 0$. It follows that λ_* is the solution of the equation

$$p - \sigma^2\lambda - \frac{1}{\sqrt{\gamma^2 - 2\lambda}} = 0,$$

The rate of exponential convergence is then given by

$$k = \Phi(\lambda_*) = p\lambda_* - \frac{1}{2}\sigma^2\lambda_*^2 - \gamma + \sqrt{\gamma^2 - 2\lambda_*}.$$

We set $\gamma = 1$, $\alpha = 1/2$, $\beta = 1/2$, to match the first moment of the liabilities in both risk model at time $t = 1$. The premium rate is then given by

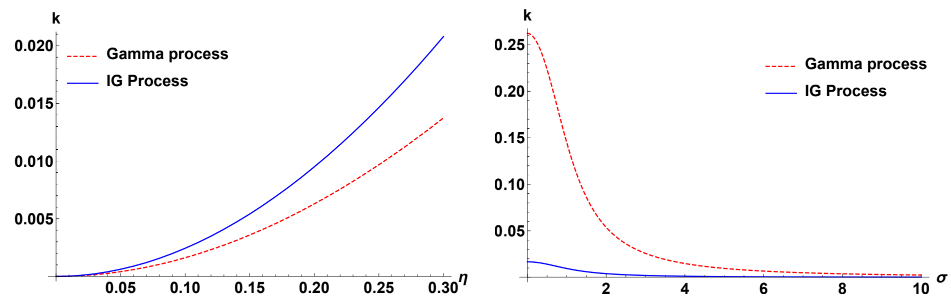
$$p = (1+\eta)\mathbb{E}(L(1)) = (1+\eta)\frac{\alpha}{\beta} = (1+\eta)\frac{1}{\gamma}. \tag{32}$$

Table 3 contains the value of the exponential rate of convergence when the liability of the insurance company is governed by a *gamma process* or an *inverse Gaussian Lévy process* depending on the safety loading and the volatility of the diffusion. Figure

		Lévy processes	
Volatility	Safety Loadings	GammaP(1/2,1/2)	IGP(1)
$\sigma = 0$	$\eta =$ 0.1	0.02617	0.05
	0.2	0.05442	0.1
	0.3	0.08441	0.15
$\sigma = 1$	$\eta =$ 0.1	0.01809	0.0271
	0.2	0.03882	0.05806
	0.3	0.06189	0.09238
$\sigma = 2$	$\eta =$ 0.1	0.00921	0.01104
	0.2	0.02013	0.02412
	0.3	0.03272	0.03923
$\sigma = 3$	$\eta =$ 0.1	0.00503	0.00552
	0.2	0.01101	0.01207
	0.3	0.01794	0.01965
$\sigma = 4$	$\eta =$ 0.1	0.00307	0.00324
	0.2	0.00671	0.00709
	0.3	0.01094	0.01153
$\sigma = 5$	$\eta =$ 0.1	0.00204	0.00212
	0.2	0.00447	0.00463
	0.3	0.00727	0.00753
$\sigma = 6$	$\eta =$ 0.1	0.00145	0.00149
	0.2	0.00317	0.00325
	0.3	0.00516	0.00529
$\sigma = 7$	$\eta =$ 0.1	0.00108	0.0011
	0.2	0.00236	0.0024
	0.3	0.00384	0.00391
$\sigma = 8$	$\eta =$ 0.1	0.00083	0.00085
	0.2	0.00182	0.00185
	0.3	0.00296	0.00301
$\sigma = 9$	$\eta =$ 0.1	0.00066	0.00067
	0.2	0.00145	0.00146
	0.3	0.00236	0.00238
$\sigma = 10$	$\eta =$ 0.1	0.00054	0.00054
	0.2	0.00118	0.00119
	0.3	0.00192	0.00193

TABLE 3: Rate of exponential convergence in Lévy driven risk models.

3(a) displays the rates of exponential convergence for the considered Lévy driven risk models. We observe that the impact of the volatility and the safety loading on



(a) The rate of exponential convergence depending on the safety loading, and volatility $\sigma = 1$.

(b) The rate of exponential convergence depending on the volatility and safety loading $\eta = 0.2$.

FIGURE 3: The rate of exponential convergence for Lévy driven risk processes.

the convergence rate remains the same as in the compound Poisson case. The rate of exponential convergence is noticeably greater when the liability of the insurance company follows an *inverse Gaussian Lévy process*.

5. Proof of Theorem 3.1

If Y was a reflected jump-diffusion with a.s. finitely many jumps in finite time, and with positive diffusion coefficient, then we could directly apply [30, Theorem 4.1, Theorem 4.3], and complete the proof of Theorem 3.1. We can have: (a) $\sigma(x) = 0$ for some x; (b) infinite Lévy measure μ.

In the proof of [30, Theorem 3.2], we used the following property: for all $t > 0$, $x \in \mathbb{R}_+$, and $A \subseteq \mathbb{R}_+$ of positive Lebesgue measure, we have $Q^t(x, A) > 0$. This property might not hold for the case $\sigma(x) = 0$ for some $x \in \mathbb{R}_+$. We bypass this difficulty by approximating the reflected jump-diffusion Y by a "regular" reflected jump-diffusion, where $\sigma(x) > 0$ for $x \in \mathbb{R}_+$, and the Lévy measure is finite.

For an $\varepsilon > 0$, let $Y_\varepsilon = (Y_\varepsilon(t),\, t \ge 0)$ be the reflected jump-diffusion on $\mathbb{R}_+$, with drift coefficient p_*, diffusion coefficient $\sigma_\varepsilon(\cdot) = \sigma(\cdot) + \varepsilon$, jump measure $\mu_\varepsilon(\cdot) = \mu(\cdot \cap [\varepsilon, \varepsilon^{-1}])$. Note that this is a reflected jump-diffusion with positive diffusion coefficient $\sigma_\varepsilon(y) > 0$ for all $y \in \mathbb{R}_+$, and with finite Lévy measure $\mu_\varepsilon(\mathbb{R}_+) < \infty$. Therefore, we can apply the results of [30] to this process. For $x \in \mathbb{R}_+$, let

$$\varphi_\varepsilon(x, \lambda) := p_*(x)\lambda + \frac{1}{2}\sigma_\varepsilon^2(x)\lambda^2 + \int_\varepsilon^{\varepsilon^{-1}} \left(e^{\lambda y} - 1\right) \mu_\varepsilon(\mathrm{d}y).$$

For every $x \ge 0$, we have that

$$\varphi(x, \lambda) - \varphi_\varepsilon(x, \lambda) = -\left[\varepsilon\sigma_\varepsilon(x) + \frac{1}{2}\varepsilon^2\right]\lambda^2 + \left(\int_0^\varepsilon + \int_{\varepsilon^{-1}}^\infty\right)\left(e^{\lambda y} - 1\right) \mu(\mathrm{d}y). \tag{33}$$

Recall also that
$$\int_0^\infty \left(e^{\lambda y} - 1\right)\, \mu(\mathrm{d}y) < \infty. \tag{34}$$
Combining (33) with (34) and the boundedness of σ from Assumption 1, we have:
$$\sup_{x\ge 0} |\varphi_\varepsilon(x,\lambda) - \varphi(x,\lambda)| \to 0,\ \varepsilon \downarrow 0. \tag{35}$$
By our assumptions, $\sup_{x\ge 0}\varphi(x,\lambda) = -\Phi(\lambda) < 0$. From (35),
$$-\sup_{x\ge 0}\varphi_\varepsilon(x,\lambda) =: \Phi_\varepsilon(\lambda) \to \Phi(\lambda).$$
Thus we conclude that there exists an $\varepsilon_0 > 0$ such that for $\varepsilon \in [0, \varepsilon_0]$, $\Phi_\varepsilon(\lambda) > 0$. Apply [30, Theorem 4.3] to prove the statement of Theorem 3.1 for the process Y_ε. For consistency of notation, denote $Y_0 := Y$. There exists a unique stationary distribution π_ε for Y_ε, which satisfies $(\pi_\varepsilon, V_\lambda) < \infty$; its transition kernel $Q^t_\varepsilon(x,\cdot)$ satisfies
$$\|Q^t_\varepsilon(x,\cdot) - \pi_\varepsilon(\cdot)\|_{V_\lambda} \le [V_\lambda(x) + (\pi_\varepsilon, V_\lambda)]\, e^{-\Phi_\varepsilon(\lambda)t}. \tag{36}$$
We would like to take the limit $\varepsilon \downarrow 0$ in (36). To this end, let us introduce some new notation. Take a smooth nondecreasing C^∞ function $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ with
$$\theta(x) = \begin{cases} 0,\ x \le s_-; \\ x,\ x \ge s_+; \end{cases} \qquad \theta(x) \le x,$$
for some fixed $s_+ > s_- > 0$. The function θ is Lipschitz on $\mathbb{R}_+$: there exists a constant $C(\theta) > 0$ such that
$$|\theta(s_1) - \theta(s_2)| \le C(\theta)|s_1 - s_2| \text{ for all } s_1, s_2 \in \mathbb{R}_+. \tag{37}$$
Next, define $\tilde V_\lambda(x) = V_\lambda(\theta(x)) = e^{\lambda\theta(x)}$. The process Y_ε has the generator $\mathcal{L}_\varepsilon$:
$$\mathcal{L}_\varepsilon f(x) = p_*(x) f'(x) + \frac{1}{2}\sigma^2_\varepsilon(x) f''(x) + \int_\varepsilon^{\varepsilon^{-1}} [f(x+y) - f(x)]\, \mu(\mathrm{d}y),$$
for $f \in C^2(\mathbb{R}_+)$ with $f'(0) = 0$. Repeating calculations from [30, Theorem 3.2] with minor changes, we get:
$$\begin{aligned} \mathcal{L}_\varepsilon \tilde V_\lambda(x) &\le -\Phi_\varepsilon(\lambda)\tilde V_\lambda(x) + c_\varepsilon 1_{[0,s_+]}(x),\ x \in \mathbb{R}_+, \\ c_\varepsilon &:= \max_{x\in[0,s_+]} \left[\mathcal{L}_\varepsilon \tilde V_\lambda(x) + \varphi_\varepsilon(\lambda, x)\tilde V_\lambda(x)\right]. \end{aligned} \tag{38}$$

Lemma 7. *$\overline{\lim}_{\varepsilon\downarrow 0}(\pi_\varepsilon, V_\lambda) < \infty$.*

Proof. The functions V_λ and $\tilde V_\lambda(x)$ are of the same order, in the sense that
$$0 < \inf_{x\ge 0}\frac{\tilde V_\lambda(x)}{V_\lambda(x)} \le \sup_{x\ge 0}\frac{\tilde V_\lambda(x)}{V_\lambda(x)} < \infty. \tag{39}$$

Therefore, it suffices to show that

$$\overline{\lim_{\varepsilon\downarrow 0}}(\pi_\varepsilon, \tilde{V}_\lambda) < \infty. \tag{40}$$

Apply the measure π_ε to both sides of the inequality (38). This probability measure is stationary; therefore, the left-hand side of (38) becomes $(\pi_\varepsilon, \mathcal{L}_\varepsilon\tilde{V}_\lambda) = 0$. Therefore, $-\Phi_\varepsilon(\lambda)(\pi_\varepsilon, \tilde{V}_\lambda) + c_\varepsilon(\pi_\varepsilon, 1_{[0,s_+]}) \ge 0$. From $(\pi_\varepsilon, 1_{[0,s_+]}) = \pi_\varepsilon([0, s_+]) \le 1$, we get:

$$\left(\pi_\varepsilon, \tilde{V}_\lambda\right) \le \frac{c_\varepsilon}{\Phi_\varepsilon(\lambda)}. \tag{41}$$

By (41), to show (40), it suffices to show that $\overline{\lim}_{\varepsilon\downarrow 0} c_\varepsilon < \infty$. This, in turn, would follow from (38), and the following relation:

$$\mathcal{L}_\varepsilon\tilde{V}_\lambda(x) \to \mathcal{L}\tilde{V}_\lambda(x), \quad \text{uniformly on} \quad [0, s_+]. \tag{42}$$

We can express the difference of generators as

$$\begin{aligned}
&\mathcal{L}_\varepsilon\tilde{V}_\lambda(x) - \mathcal{L}\tilde{V}_\lambda(x)\\
&\quad= \frac{1}{2}\left(\sigma_\varepsilon^2(x) - \sigma^2(x)\right) f''(x) - \left(\int_0^\varepsilon + \int_{\varepsilon^{-1}}^\infty\right)\left[\tilde{V}_\lambda(x+y) - \tilde{V}_\lambda(x)\right]\,\mu(\mathrm{d}y).
\end{aligned} \tag{43}$$

The first term in the right-hand side of (43) is equal to $\frac{1}{2}(2\varepsilon\sigma(x) + \varepsilon^2)f''(x)$. Since σ is bounded, this term converges to 0 as $\varepsilon\downarrow 0$ uniformly on $[0, s_+]$. It suffices to prove that the second term converges to zero as well. For all $x, y \ge 0$, using (37), we have:

$$\begin{aligned}
0 &\le \tilde{V}_\lambda(x+y) - \tilde{V}_\lambda(x) = e^{\lambda\theta(x+y)} - e^{\lambda\theta(x)}\\
&= e^{\lambda\theta(x)}\left[e^{\lambda(\theta(x+y)-\theta(x))} - 1\right] \le \tilde{V}_\lambda(x)\left[e^{\lambda C(\theta)y} - 1\right].
\end{aligned} \tag{44}$$

Changing the parameter s_- and letting $s_-\downarrow 0$, we have: $\theta(x) \to x$ uniformly on $\mathbb{R}_+$. Therefore, we can make the Lipschitz constant $C(\theta)$ as close to 1 as necessary. Also, note that for λ' in some neighborhood of λ, we have:

$$\int_0^\infty \left(e^{\lambda' x} - 1\right)\,\mu(\mathrm{d}x) < \infty. \tag{45}$$

Combining (44), (45), using that $\sup_{x\in[0,s_+]}\tilde{V}_\lambda(x) < \infty$, taking $C(\theta)$ close enough to 1, we complete the proof that the second term in the right-hand side of (43) tends to 0 as $\varepsilon\downarrow 0$. This completes the proof of (42), and with it that of Lemma 7. □

Now, we state a fundamental lemma, and complete the proof of Theorem 3.1 assuming that this lemma is proved. The proof is postponed until Appendix.

Lemma 8. *Take a version $\tilde{Y}_\varepsilon$ of the reflected jump-diffusion Y_ε, starting from $y_\varepsilon \ge 0$, for $\varepsilon \ge 0$. If $y_\varepsilon \to y_0$, then we can couple $\tilde{Y}_\varepsilon$ and $\tilde{Y}_0$ so that for every $T \ge 0$,*

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\sup_{0\le t\le T}\left|\tilde{Y}_\varepsilon(t) - \tilde{Y}_0(t)\right|^2 = 0.$$

Lemma 9. *The reflected jump-diffusion Y_0 has a stationary distribution π_0.*

Proof. Since $V_\lambda(\infty) = \infty$, Lemma 7 implies tightness of the familly $(\pi_\varepsilon)_{\varepsilon\in(0,\varepsilon_0]}$ of probability measures. Now take a stationary version $\overline{Y}_\varepsilon$ of the reflected jump-diffusion Y_ε: for every $t \ge 0$, let $\overline{Y}_\varepsilon(t) \sim \pi_\varepsilon$. Take a sequence $(\varepsilon_n)_{n\ge1}$ such that $\varepsilon_n \downarrow 0$ as $n \to \infty$, and $\pi_{\varepsilon_n} \Rightarrow \pi_0$ (where $\Rightarrow$ stands for weak convergence) for some probability measure π_0 on $\mathbb{R}_+$. It follows from Lemma 8 that for every $t \ge 0$, we have: $\overline{Y}_{\varepsilon_n}(t) \Rightarrow \overline{Y}_0(t)$ as $n \to \infty$, where $\overline{Y}_0$ is a stationary version of the reflected jump-diffusion Y_0: that is, $\overline{Y}_0(t) \sim \pi_0$ for every $t \ge 0$. □

Next, take a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$ such that $|g(x)| \le V_\lambda(x)$, $x \in \mathbb{R}_+$.

Lemma 10. *$(\pi_{\varepsilon_n}, g) \to (\pi_0, g)$ as $n \to \infty$.*

Proof. The function Φ is a supremum of a family of functions $-\varphi(\cdot, x)$, which are continuous in λ. Therefore, Φ is lower semicontinuous, and the set $\{\lambda > 0 \mid \Phi(\lambda) > 0\}$ is open. Apply Lemma 7 to some $\lambda' > \lambda$ (which exists by the observation above). Then we get:

$$\overline{\lim_{\varepsilon\downarrow0}}\,(\pi_{\varepsilon_n}, V_{\lambda'}) < \infty.$$

Note also that $|g(x)|^{\lambda'/\lambda} \le [V_\lambda(x)]^{\lambda'/\lambda} = V_{\lambda'}(x)$ for all $x \ge 0$. Therefore, the family $(\pi_\varepsilon g^{-1})_{\varepsilon\in(0,\varepsilon_0]}$ of probability distributions is uniformly integrable. Uniform integrability plus a.s. convergence imply convergence of expected values. Thus we complete the proof of Lemma 10. □

For all $\varepsilon \ge 0$, take a copy Y^ε of Y_ε starting from the same initial point $x \in \mathbb{R}_+$.

Lemma 11. *For every $t \ge 0$, we have: $\mathbb{E}g(Y^\varepsilon(t)) \to \mathbb{E}g(Y^0(t))$ as $\varepsilon \downarrow 0$.*

Proof. Following calculations in the proof of [30, Theorem 3.2], we get:

$$\mathbb{E}\tilde{V}_\lambda(Y^\varepsilon(t)) - \tilde{V}_\lambda(x) \le \int_0^t \left[-\Phi_\varepsilon(\lambda)\tilde{V}_\lambda(Y^\varepsilon(s)) + c_\varepsilon 1_{[0,s_+]}(s)\right] \mathrm{d}s \le c_\varepsilon t. \tag{46}$$

Therefore, from (46) we have: $\overline{\lim}_{\varepsilon\downarrow0}\,\mathbb{E}\tilde{V}_\lambda(Y^\varepsilon(t)) < \infty$. Because of (39), this holds for V_λ in place of $\tilde{V}_\lambda$. This is also true for $\lambda' > \lambda$ slightly larger than λ. Applying the same uniform integrability argument as in the proof of Lemma 10, we complete the proof of Lemma 11. □

Finally, let us complete the proof of Theorem 3.1. From (36), we have:

$$|\mathbb{E}g(Y^\varepsilon(t)) - (\pi_\varepsilon, g)| \le [V_\lambda(x) + (\pi_\varepsilon, V_\lambda)]\, e^{-\Phi_\varepsilon(\lambda)t}. \tag{47}$$

Taking $\varepsilon = \varepsilon_n$ and letting $n \to \infty$ in (47), we use Lemma 10 and 11 to conclude that

$$\left|\mathbb{E}g(Y^0(t)) - (\pi_0, g)\right| \le [V_\lambda(x) + (\pi_0, V_\lambda)]\, e^{-\Phi(\lambda)t}. \tag{48}$$

Take the supremum over all functions $g : \mathbb{R}_+ \to \mathbb{R}$ which satisfy $|g(x)| \le V_\lambda(x)$ for all $x \in \mathbb{R}_+$, and complete the proof of Theorem 3.1 for Lipschitz p_*.

6. Concluding Remarks

We showed that the convergence of ruin probabilities in a rather broad class of risk processes is achieved exponentially fast. This rate is easy to compute (at least in the examples considered in Section 4), and happened to be sharp when the premium rate and its variability are independent from the current wealth of the insurance company.

A natural question relies on the practical implication of having access to the value of the rate of exponential convergence; in particular, whether this leads to an numerical approximation of the finite time ruin probability. This issue has been discussed in Asmussen [2], the answer was negative. For constant premium rate and diffusion parameter, one may approximate the actual gap between the ruin probabilities using numerical integration techniques based on Michna et al [23].

Another direction is to relax the condition upon the tail of the claim size. It is of practical interest to let the claim size distribution be heavy tailed. An extension of the early work of Asmussen and Teugels [4] could be envisaged. For example, in the work of Tang [35], a compound Poisson risk model under constant interest force with sub-exponentially distributed claim size is considered. When comparing the asymptotics provided by Tang [35, (2.5), (3.2)], it seems that exponential convergence holds for large initial reserves.

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Appendix A. Proof of Lemma 1

As a preliminary step, we claim that

$$\mathbb{E}\exp\left[\lambda_0 \sup_{0\le s\le t} L(s)\right] = \mathbb{E}\left[e^{\lambda_0 L(t)}\right] < \infty. \tag{49}$$

This follows from plugging $\lambda := -\lambda_0$ into (9) and from monotonicity of L. Turning to the proof of the main statement, we first claim that for all functions $f \in C^\infty(\mathbb{R})$ which satisfy (12), the function $\mathcal{N}f$ is well-defined, and satisfies for some constant D_f:

$$|\mathcal{N}f(x)| \le D_f e^{\lambda_0 x},\ x \ge 0, \tag{50}$$

Indeed, for $y \in [0,1]$ and $x \ge 0$, $|f(x+y) - f(x)| \le \sup|f'| \cdot y$. Thus

$$\int_0^1 (f(x+y) - f(x))\,\mathrm{d}\mu(y) \le \sup|f'| \cdot \int_0^1 y\,\mathrm{d}\mu(y) < \infty. \tag{51}$$

But for $y \ge 1$ we have:

$$\begin{aligned}\Big|\int_1^{\infty} [f(x+y) - f(x)]\, \mu(\mathrm{d}y)\Big| &\le f(x)\mu[1,\infty) + C_f e^{\lambda_0 x}\int_1^{\infty} e^{\lambda_0 y}\,\mu(\mathrm{d}y) \\ &\le C_f e^{\lambda_0 x}\left[\mu[1,\infty) + \int_1^{\infty} e^{\lambda_0 y}\,\mu(\mathrm{d}y)\right].\end{aligned} \tag{52}$$

Combining (51) and (52), we get (50) for

$$D_f := C_f\left[\mu[1,\infty) + \int_1^{\infty} e^{\lambda_0 y}\,\mu(\mathrm{d}y)\right] + \sup|f'|\cdot\int_0^1 y\,\mathrm{d}\mu(y).$$

Combining (49) with (50), we get:

$$\mathbb{E}\,[f(L(t))| < \infty, \quad \sup_{0\le s\le t}\mathbb{E}\,|(\mathcal{N}f)(L(s))| < \infty, \quad t > 0. \tag{53}$$

By the standard stopping argument, the following process is a local martingale:

$$f(L(t)) - f(L(0)) - \int_0^t (\mathcal{N}f)(L(s))\,\mathrm{d}s,\ t \ge 0. \tag{54}$$

Using (53) we get that the process (54) is an actual martingale. Taking expectation (if $x := L(0)$ is a constant), we get:

$$\mathbb{E}\,[f(L(t))] - f(x) - \int_0^t \mathbb{E}\,[(\mathcal{N}f)(L(s))]\,\mathrm{d}s = 0.$$

Dividing by t and letting $t \downarrow 0$, we get:

$$\lim_{t\downarrow 0}\frac{1}{t}\left(\mathbb{E}[f(L(t))] - f(x)\right) = (\mathcal{N}f)(x).$$

Appendix B. Proof of Lemma 2

From Assumption 3 it follows that $\int_1^{\infty} x\,\mu(\mathrm{d}x) < \infty$, and from (8) we conclude that $\int_0^1 x\,\mu(\mathrm{d}x) < \infty$. Condition (14) then immediately follows from these two observations.

Appendix C. Proof of Lemma 3

The proof is very similar to the classic existence and uniqueness theorem for an SDE, which can be found in any stochastic analysis textbook, done by Picard iterations or fixed point theorem. But for the sake for completeness, we provide the complete proof.

Take every filtered probability space with two independent processes: a Brownian motion W, and a Lévy process L. Fix time horizon $T > 0$, and consider the space $\mathcal{E}_T$ of all right-continuous adapted processes $Z = (Z(t),\, 0 \le t \le T)$ with left limits and

$$\|Z\|_{2,T}^2 := \mathbb{E}\sup_{0\le t\le T} Z^2(t) < \infty. \tag{55}$$

This is a Banach space with norm $\|\cdot\|_{2,T}$. Let us introduce a mapping $\mathcal{P} : \mathcal{E}_T \to \mathcal{E}_T$:

$$\mathcal{P}(Z)(t) = u + \int_0^t p_*(Z(s))\,\mathrm{d}s + \int_0^t \sigma(Z(s))\,\mathrm{d}W(s) - L(t),\ 0 \le t \le T.$$

Any solution to the stochastic integral equation (7) is a fixed point for this mapping $\mathcal{P}$. It is straightforward to show, using Lipschitz properties of p_* and σ, that these mappings indeed map $\mathcal{E}_T$ into $\mathcal{E}_T$. Assume $C(p_*)$ and $C(\sigma)$ are Lipschitz constants for functions p_* and σ respectively.

Lemma 12. *For $\mathcal{Z}, \mathcal{Z}' \in \mathcal{E}_T$, the following Lipschitz property holds:*

$$\begin{aligned} \|\mathcal{R}(\mathcal{Z}) - \mathcal{R}(\mathcal{Z}')\|_{2,T} \le C_T \|\mathcal{Z} - \mathcal{Z}'\|_{2,T}; \\ C_T := C(p_*)T + 2C(\sigma)T^{1/2}. \end{aligned} \tag{56}$$

Proof. For $t \in [0, T]$, we can represent the difference $\mathcal{P}(\mathcal{Z})(t) - \mathcal{P}(\mathcal{Z}')(t)$ as

$$\int_0^t [p_*(\mathcal{Z}(s)) - p_*(\mathcal{Z}'(s))]\ \mathrm{d}s + \int_0^t [\sigma(\mathcal{Z}(s)) - \sigma(\mathcal{Z}'(s))]\ \mathrm{d}W(s) =: I(t) + M(t). \tag{57}$$

Apply the norm $\|\cdot\|_{T,2}$ to (57):

$$\|\mathcal{P}(\mathcal{Z})(t) - \mathcal{P}(\mathcal{Z}')(t)\|_{2,T} \le \|I(t)\|_{2,T} + \|M(t)\|_{2,T}. \tag{58}$$

The norm $\|I(t)\|_{2,T}$ is estimated using the Lipschitz property of σ:

$$\begin{aligned} \|I(t)\|^2_{2,T} &= \mathbb{E} \sup_{0\le t\le T} I^2(t) \le \mathbb{E} \sup_{0\le t\le T} \left(\int_0^t C(p_*)\,[\mathcal{Z}(s) - \mathcal{Z}'(s)]\ \mathrm{d}s \right)^2 \\ &\le T^2 C^2(p_*) \cdot \mathbb{E} \sup_{0\le s\le T} [\mathcal{Z}(s) - \mathcal{Z}'(s)]^2 = T^2 C^2(p_*) \|\mathcal{Z} - \mathcal{Z}'\|^2_{2,T}. \end{aligned} \tag{59}$$

Finally, the norm $\|M\|_{2,T}$ can be estimated using the martingale inequalities:

$$\begin{aligned} \|M\|^2_{2,T} &= \mathbb{E} \sup_{0\le t\le T} M^2(t) \le 4\mathbb{E}M^2(T) = 4\int_0^T [\sigma(\mathcal{Z}(s)) - \sigma(\mathcal{Z}'(s))]^2\ \mathrm{d}s \\ &\le 4C^2(\sigma)T \cdot \mathbb{E} \sup_{0\le t\le T} (\mathcal{Z}(t) - \mathcal{Z}'(t))^2 = 4C^2(\sigma)T\|\mathcal{Z} - \mathcal{Z}'\|^2_{2,T}. \end{aligned} \tag{60}$$

Combining (58), (59), (60), we complete the proof of (56). □

For small enough T, the constant C_T from (71) is strictly less than 1. Assume this is the case until the end of the proof. Then the mapping $\mathcal{R}$ is contractive. Therefore, it has a unique fixed point, which can be obtained by successive approximations:

$$\mathcal{Y} = \lim_{n\to\infty} \mathcal{R}^n(\mathcal{Z}) \quad \text{for every} \quad \mathcal{Z} \in \mathcal{E}_T.$$

This completes the proof of Lemma 3 for small enough T. Thus we can prove strong existence and pathwise uniqueness on the time interval $[0, T]$, and then on $[T, 2T]$,

$[2T, 3T]$, etc. The formula for the generator in (15) then follows from straightforward application of Itô's formula.

Appendix D. Proof of Lemma 4

Similar to the proof of Lemma 4, but we have a Skorohod mapping $\mathcal{S}$:

$$\mathcal{S}(Z)(t) = Z(t) + \sup_{0 \le s \le t} (Z(s))_-, \quad 0 \le t \le T,$$

where $(a)_- := \max(-a, 0)$ for any $a \in \mathbb{R}$. Thus any solution to the stochastic integral equation (16) defining the reflected process is a fixed point mapping of $\mathcal{S} \circ \mathcal{R}$ in the space $\mathcal{E}_T$ from (55). But this mapping $\mathcal{S}$ is 1-Lipschitz, see [36], and thus for small T the mapping $\mathcal{S} \circ \mathcal{R}$ is contractive. Therefore the fixed point exists and is unique by the classic theorem. Similarly to the proof of Lemma 4, the formula for the generator in (17) follows by Itô's formula.

Appendix E. Proof of Lemma 5

Consider two copies X_1 and X_2 of this process, starting from $X_1(0) = x_1$ and $X_2(0) = x_2$, where $x_1 > x_2 \ge 0$. Let us couple them: that is, we create their copies on a common probability space, using the same driving Brownian motion W and Lévy process L. We can do this by Lemma 3. Next, we aim to prove that $X_1(t) \ge X_2(t)$ for all $t \ge 0$ simultaneously, with probability 1. This would automatically imply that $\mathbb{P}(X_1(t) \ge c) \ge \mathbb{P}(X_2(t) \ge c)$ for all $t, c \ge 0$, which is the property (b) in Theorem 1.

Assume there exists a $t > 0$ such that $X_1(t) < X_2(t)$. Let $\tau := \inf\{t \ge 0 \mid X_1(t) < X_2(t)\}$. By right-continuity of X_1 and X_2, we must have $X_1(\tau) \le X_2(\tau)$. But we cannot have $X_1(\tau) = X_2(\tau)$, because then by strong Markov property we would have $X_1(t) = X_2(t)$ for all $t \ge \tau$ (recall that τ is a stopping time). Therefore,

$$X_1(\tau) < X_2(\tau), \text{ but } X_1(\tau-) \ge X_2(\tau-). \tag{61}$$

Thus, τ is a jump time for both X_1 and X_2, that is, for the Lévy process L. The displacement during the jump must be the same for X_1 and X_2:

$$X_1(\tau) - X_1(\tau-) = -\left[L(\tau) - L(\tau-)\right] = X_2(\tau) - X_2(\tau-). \tag{62}$$

The contradiction between (61) and (62) completes the proof of Lemma 5.

Appendix F. Proof of Lemma 6

We can adapt the proof of [33, Proposition 4.3] to the case with jumps. We also point the reader to the construction in [10, Example 4.1], used to establish the duality between an absorbed and a reflected random walk. In the notation of [33, Proposition 4.3], we change the following: L is a two-sided Lévy process, that is, $(L(t)-L(0),\, t \ge 0)$ and $(L(0) - L(-t),\, t \ge 0)$ are i.i.d. Lévy processes; $(B_r,\, r \ge 0)$ a two-sided Brownian motion, and $\tilde{B}_r := B_{-r}$, $r \ge 0$; and we define the reflected process as

$$
\begin{aligned}
Y_s[y] &= y + \int_0^s p_*(V_r[y])\,\mathrm{d}r + \int_0^s \alpha(V_r[y])\,\mathrm{d}B_r + L(s),\ \ s, y \ge 0;\\
V_t[y] &= Y_t[y] + \sup_{0\le s\le t} (Y_s[y])_-.
\end{aligned}
\tag{63}
$$

We have other notation from [10]:

$$
\begin{aligned}
X_s^{(-t)}[y] &= y + \int_{-s}^t p_*(X_r[y])\,\mathrm{d}r + \int_0^s \alpha(X_r[y])\,\mathrm{d}B_r + L(s) - L(-t),\ \ s, y \ge 0;\\
W_t[x] &= x + \int_0^t p(W_s[x])\,\mathrm{d}s + \int_0^t \alpha(W_s[x])\,\mathrm{d}\tilde{B}_s - L(t),\ \ t, x \ge 0.
\end{aligned}
\tag{64}
$$

In this article, the authors denote the driving Brownian motion inside the Itô integral by B, not W. In (64), the process W stands for the risk process, which we in our article denote by X. The process Y defined in the system (63) is the reflected jump-diffusion, which we also denote in our article by Y. The first equation in (64) defines the non-reflected version of the process Y, starting from time $-t$.

The proof can be adapted *mutatis mutandis*, with the following corrections: We use right-continuous versions with left limits instead of left-continuous versions. (We refer the reader to [33, p.431, Example 1].) The statement at the top of [33, p.438] that for every trajectory of B and L, and every $T > 0$, there exists an $x' > 0$ such that $W_r(x') > 0$ for all $r \in [0, T]$. In [33], this follows from the continuity of $X_s[y]$ in s. Now we no longer have this property. But in our case, this follows from the following comparison: Let $L(T) = x''$. Take a copy of $\tilde{W}_t[x']$ without jumps, which we start from x':

$$
\tilde{W}_t[x'] = x + \int_0^t p(\tilde{W}_s[x'])\,\mathrm{d}s + \int_0^t \alpha(\tilde{W}_s[x'])\,\mathrm{d}B_s.
$$

Take an $x' > 0$ such that $\tilde{W}_t[x'] > 0$ for all $t \in [0, T]$. Then the copy of $W_t[x' + x'']$ starting from $x' + x''$ is estimated from below by $\tilde{W}_t[x']$. This follows from the standard comparison arguments. Thus $W_t[x' + x''] > 0$ for all $t \in [0, T]$.

Appendix G. Proof of Lemma 8

This is similar to the proof of Lemma 4, but we have to add a few details. Let us take a probability space with independent Brownian motion W and Lévy process L, and let L_ε be a subordinator process with Lévy measure μ_ε, obtained from L by eliminating all jumps of size less than ε and greater than ε^{-1}. For consistency of notation, let $L_0 := 0$. For every $\varepsilon \ge 0$, we can represent

$$
\tilde{Y}_\varepsilon(t) = y_\varepsilon + \int_0^t p_*(\tilde{Y}_\varepsilon(s))\,\mathrm{d}s + \int_0^t \sigma_\varepsilon(\tilde{Y}_\varepsilon(s))\,\mathrm{d}W(s) + L_\varepsilon(t) + N_\varepsilon(t),\ \ t \ge 0.
\tag{65}
$$

Here, N_ε is a nondecreasing right-continuous process with left limits, with $N_\varepsilon(0) = 0$, which can increase only when $\tilde{Y}_\varepsilon = 0$. We can rewrite (65) as

$$\tilde{Y}_\varepsilon(t) = \mathcal{X}_\varepsilon(t) + \int_0^t p_*(\tilde{Y}_\varepsilon(s))\,\mathrm{d}s + \int_0^t \sigma(\tilde{Y}_\varepsilon(s))\,\mathrm{d}W(s) + N_\varepsilon(t),\ t \ge 0. \tag{66}$$

Here, we introduce a new piece of notation:

$$\mathcal{X}_\varepsilon(t) = y_\varepsilon + L_\varepsilon(t) + \varepsilon W(t),\ t \ge 0. \tag{67}$$

The process $L(\cdot) - L_\varepsilon(\cdot)$ is nondecreasing. By Assumption 3 as $\varepsilon \downarrow 0$, for every $T > 0$,

$$\mathbb{E}\sup_{0\le t\le T}|L(t) - L_\varepsilon(t)|^2 = \mathbb{E}\left(L(T) - L_\varepsilon(T)\right)^2 = T\left(\int_0^\varepsilon + \int_{\varepsilon^{-1}}^\infty\right) x^2\,\mu(\mathrm{d}x) \to 0. \tag{68}$$

From (67) and (68), we have:

$$\mathbb{E}\sup_{0\le t\le T}|\mathcal{X}_0(t) - \mathcal{X}_\varepsilon(t)|^2 \to 0,\ \varepsilon \downarrow 0. \tag{69}$$

Fix time horizon $T > 0$, and consider the space $\mathcal{E}_T$ as above. Fix an $\mathcal{X} \in \mathcal{E}_T$. Let us introduce a mapping $\mathcal{P}_{\mathcal{X}}$:

$$\mathcal{P}_{\mathcal{X}}(Z)(t) = \mathcal{X}(t) + \int_0^t p_*(Z(s))\,\mathrm{d}s + \int_0^t \sigma(Z(s))\,\mathrm{d}W(s),\ 0 \le t \le T.$$

For any $\mathcal{X} \in \mathcal{E}_T$, let $\mathcal{R}_{\mathcal{X}} := \mathcal{S} \circ \mathcal{P}_{\mathcal{X}}$. Then we can represent (66) as

$$Y_\varepsilon = (\mathcal{S} \circ \mathcal{P}_{\mathcal{X}_\varepsilon}(Y_\varepsilon)) = \mathcal{R}_{\mathcal{X}_\varepsilon}(Y_\varepsilon). \tag{70}$$

As before, these mappings indeed map $\mathcal{E}_T$ into $\mathcal{E}_T$. Similarly to the proof of Lemma 12, we can prove the following result:

Lemma 13. *For $\mathcal{X}, \mathcal{X}', \mathcal{Z}, \mathcal{Z}' \in \mathcal{E}_T$, the following holds with constant C_T from* (56)*:*

$$\|\mathcal{R}_{\mathcal{X}}(\mathcal{Z}) - \mathcal{R}_{\mathcal{X}'}(\mathcal{Z}')\|_{2,T} \le C_T\|\mathcal{Z} - \mathcal{Z}'\|_{2,T} + \|\mathcal{X} - \mathcal{X}'\|_{2,T}. \tag{71}$$

Proof. Since $\mathcal{S}$ is 1-Lipschitz, it suffices to show (71) for $\mathcal{P}_{\mathcal{X}}$ instead of $\mathcal{R}_{\mathcal{X}}$. We can express the difference between $\mathcal{P}_{\mathcal{X}}(\mathcal{Z})$ and $\mathcal{P}_{\mathcal{X}'}(\mathcal{Z}')$ as follows: for $t \in [0, T]$,

$$\begin{aligned}\mathcal{P}_{\mathcal{X}}(\mathcal{Z})(t) - \mathcal{P}_{\mathcal{X}'}(\mathcal{Z}')(t) &= \mathcal{X}(t) - \mathcal{X}'(t)\\ &+ \int_0^t \left[p_*(\mathcal{Z}(s)) - p_*(\mathcal{Z}'(s))\right]\,\mathrm{d}s + \int_0^t \left[\sigma(\mathcal{Z}(s)) - \sigma(\mathcal{Z}'(s))\right]\,\mathrm{d}W(s).\end{aligned} \tag{72}$$

Denote by I and M the second and third terms in the right-hand side of (72):

$$\|\mathcal{P}_{\mathcal{X}}(\mathcal{Z})(t) - \mathcal{P}_{\mathcal{X}'}(\mathcal{Z}')(t)\|_{2,T} \le \|\mathcal{X} - \mathcal{X}'\|_{2,T} + \|I\|_{2,T} + \|M\|_{2,T}. \tag{73}$$

The two last terms in the right-hand side of (73) are estimated as in (59) and (60). $\square$

As before, for small enough T, the constant C_T from (56) is strictly less than 1. Assume this is the case until the end of the proof. Then for every $\mathcal{X} \in \mathcal{E}_T$, the mapping $\mathcal{R}_{\mathcal{X}}$ is contractive. Therefore, it has a unique fixed point, which can be obtained by successive approximations: $\mathcal{Y}(\mathcal{X}) = \lim_{n\to\infty} \mathcal{R}^n_{\mathcal{X}}(\mathcal{Z})$. In particular, the equation (70) has a unique solution, which is obtained by successive approximations:

$$Y_\varepsilon = \lim_{n\to\infty} \mathcal{R}^n_{\mathcal{X}_\varepsilon}(\mathcal{Z}).$$

We can take $\mathcal{Z} = 0$ as initial condition, or any other element in $\mathcal{E}_T$. Applying the mappings in Lemma 13 once again, we have:

$$\|\mathcal{R}^2_{\mathcal{X}}(\mathcal{Z}) - \mathcal{R}^2_{\mathcal{X}'}(\mathcal{Z}')\|_{2,T} \le C_T^2\|\mathcal{Z} - \mathcal{Z}'\| + (1 + C_T)\|\mathcal{X} - \mathcal{X}'\|.$$

By induction over $n = 1, 2, \ldots$ we get, with $C_{T,n} := 1 + C_T + \ldots + C_T^{n-1}$:

$$\|\mathcal{R}^n_{\mathcal{X}}(\mathcal{Z}) - \mathcal{R}^n_{\mathcal{X}'}(\mathcal{Z}')\|_{2,T} \le C_T^n\|\mathcal{Z} - \mathcal{Z}'\|_{2,T} + C_{T,n}\|\mathcal{X} - \mathcal{X}'\|_{2,T}. \tag{74}$$

Let $n \to \infty$ in (74). If $C_T < 1$, then

$$\|\mathcal{Y}(\mathcal{X}) - \mathcal{Y}(\mathcal{X}')\|_{2,T} \le \frac{1}{1 - C_T}\|\mathcal{X} - \mathcal{X}'\|_{2,T}. \tag{75}$$

Letting $\mathcal{X} := \mathcal{X}_0,\ \mathcal{X}' := \mathcal{X}_\varepsilon$ in (75), and using (69), we complete the proof of Lemma 8.

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